L_p Spaces ($p < 1$)
Or
"Is there life without Hahn-Banach Theorem?"

N. Tenney Peck
University of Illinois

Fall 1980

Notes by Larry Riddle

8/25 L_p $(p<1)$ or "Is there life without Hahn-Banach theorem?"

$L_0 := \{ f : f \text{ measurable on } [0,1] \}$ with metric

$$\rho_0(f,g) = \int_0^1 \frac{|f-g|}{1+|f-g|}\, d\mu$$

$L_p := \{ f : f \text{ measurable on } [0,1],\ \int |f|^p d\mu < \infty \}$

$$\|f-g\|_p = \left(\int |f-g|^p d\mu \right)^{1/p}$$

$$\ell_p := \left\{ (a_n) : \|(a_n)\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} < \infty \right\}$$

$$H_p := \left\{ f : f \text{ analytic in } |z|<1,\ \sup_{r<1} \left(\frac{1}{2\pi} \int |f(re^{i\theta})|^p d\theta \right)^{1/p} < \infty \right\}$$

$\|f\|$

Some things are still true for F-spaces

<u>DEFINITION</u>: An F-space is a metric TVS (E,ρ) which is ρ-complete

We'll assume that $\rho(x,y) = |x-y|$ (F-norm) where

i) $|x+y| \le |x| + |y|$

ii) $|ax| \le |x|$ for $|a| \le 1$

OPEN MAPPING THEOREM: If E, F are F-spaces and $T: E \to F$ is an onto continuous linear operator, then T is open.

UNIFORM BOUNDEDNESS PRINCIPLE: If E, F are F-spaces and (T_α) is a net of continuous linear maps of E into F s.t. $(T_\alpha x)$ is bounded for all x, then

$$\lim_{x \to 0} T_\alpha(x) = 0 \quad \text{unif in } \alpha$$

CLOSED GRAPH THEOREM: If E, F are F-spaces and $T: E \to F$ is a linear map s.t. graph of T is closed in $E \times F$, then T is continuous.

Want to show L_0, L_p, ℓ_p, H_p are F-spaces

L_0/ Fact: a) if $x, y \geq 0$, then $\frac{x+y}{1+x+y} \leq \frac{x}{1+x} + \frac{y}{1+y}$.

b) $\frac{x}{1+x} \uparrow$ on $[0, \infty)$

The two conditions for F-norm $\int \frac{|f|}{1+|f|} \, d\mu$ are satisfied.

Also, (L_0, ρ_0) is complete since ρ_0-convergence is just convergence in measure, i.e.

$$\mu\{|f_n - f| > \varepsilon\} \to 0 \quad \forall \varepsilon > 0$$

FACT: If $\mu(\operatorname{supp} f) < \varepsilon$, then $\int \frac{|f|}{1+|f|}\, d\mu < \varepsilon$

PROPOSITION: $(L_0, \rho_0)^* = \{0\}$

Proof. Enough to prove that if U is any nbhd of 0, then $\operatorname{conv} U = L_0$. To see this, let $\varepsilon > 0$. Choose a partition $(A_i)_1^n$ of $[0,1]$ with $\mu(A_i) < \varepsilon \ \forall i$. If $f \in L_0$, then

$$f = \frac{1}{n} \sum_{i=1}^{n} n f \chi_{A_i} \quad \text{(convex sum)}$$

Also, for each i

$$\int \frac{|nf\chi_{A_i}|}{1+|nf\chi_{A_i}|}\, d\mu < \varepsilon$$

Hence $f \in \operatorname{conv} U$ $\quad (\operatorname{Ball}(0,\varepsilon) \subset U)$

ℓ_p/ The topology here is $|(a_n) - (b_n)| = \left(\sum_{n=1}^{\infty} |a_n - b_n|^p\right)^{1/p}$

PROPOSITION: (a) $(\ell_p)^*$ separates the points of ℓ_p
(b) ℓ_p is not locally convex for $p < 1$

Proof. (a) Note that $(a_n) \mapsto a_n$ is a linear functional and

$$|a_n| \leq \left(\sum |a_n|^p \right)^{1/p}$$

(hence continuous)

(b) Let (e_i) be the usual basis of ℓ_p $\left(e_i(n) = \delta_{n,i} \right)$

Let

$$y_n = \frac{1}{n} \left(\sum_{i=1}^{n} e_i \right)$$

Then $\|y_n\|_p = 1/n (n^{1/p}) = n^{1/p - 1} \longrightarrow \infty$. This shows that ℓ_p is not locally convex since the convex hull of things in the unit ball contains an unbounded sequence.

Facts. If x, y are in $\mathbb{R}^+$, $0 < p \leq 1$

(a) $(x+y)^p \leq x^p + y^p$

(b) $(1+x)^p \leq 1 + x^p$ (O.K. at 0, check derivatives)

This shows that $\|\cdot\|_p^p$ is subadditive

8/27 L_p $(0<p<1)$

$$\left(\sum |a_i + b_i|^p\right)^{1/p} \leq \left(\sum |a_i|^p + \sum |b_i|^p\right)^{1/p}$$
$$\leq 2^{1/p - 1}\left(\left(\sum |a_i|^p\right)^{1/p} + \left(\sum |b_i|^p\right)^{1/p}\right)$$

$\uparrow$ since $1/p > 1$

(With $1/p$ - quasinorm ; without $1/p$ F-norm
both give same topology)

<u>PROPOSITION</u>: l_p^* is isometrically and canonically l_∞

Proof. Given $x = (x_i)$ in l_∞, define T_x on l_p by

$$T_x(a) = \sum x_i a_i$$

(i) T_x is a continuous linear functional

(ii) $\|T_x\| = \sup_{\|a\|_p \leq 1} |T_x(a)| = \|x\|_\infty$

To see (i) note that

$$\left|\sum a_i x_i\right| \leq \|x\|_\infty \left|\sum a_i\right| \leq \|x\|_\infty \left|\sum |a_i|^p\right|^{1/p}$$

This also shows that $\|T_x\| \leq \|x\|_\infty$. But $T_x(e_i) = x_i$ and $\|e_i\|_p = 1$, so (ii) follows.

If f is a continuous linear functional on ℓ_p, define

$$x_i = f(e_i)$$

Then $(x_i) \in \ell_\infty$. If (a_i) is finitely non-zero in ℓ_p, then

$$f(a_i) = T_x(a)$$

$(x = (x_1, x_2, \ldots))$. Hence $f = T_x$.

L_p / $\quad \|f\|_p := \left(\int |f|^p \, d\mu \right)^{1/p}$ quasi-norm (without $1/p$ - F norm)

<u>PROPOSITION</u>: L_p is complete

Proof. Suppose (f_n) is L_p-Cauchy. Then (f_n) is measure Cauchy. So there is a measurable function f such that $f_n \to f$ in measure (and some $f_{n_k} \to f$ a.e.). Then

$$\int |f - f_n|^p \, d\mu \leq \liminf \int |f_{n_k} - f_n|^p \, d\mu \quad \forall n$$

$$< \varepsilon \quad \text{if } n \text{ is large enough}$$

PROPOSITION: $L_p^* = \{0\}$ (M. Day)

Proof. Suppose h is a non-trivial cont. linear functional. Let

$$U = \{ f \in L_p : |h(f)| < 1 \}$$

Then U is an open convex set in L_p containing 0. Suppose $f \in L_p$ and $|f| \leq M$ a.e. Take ε such that $B(0,\varepsilon) \subset U$. Let $A_1, \ldots, A_n$ be disjoint intervals of equal length which partition $[0,1]$ and $\mu(A_i) < \varepsilon$ $\forall i$. Then

$$f = \frac{1}{n} \sum_{i=1}^{n} n f \cdot \chi_{A_i} \qquad \text{(convex sum)}$$

Note that

$$\| n f \chi_{A_i} \|_p = \left(\int |n \chi_{A_i} f|^p \, d\mu \right)^{1/p} \leq nM \left(\frac{1}{n}\right)^{1/p}$$

$$= M n^{1-1/p} \longrightarrow 0 \text{ as } n \to \infty$$

So for large n, each $n \chi_{A_i} f$ is in U, and therefore f is in U. Hence $|h| < 1$ on a dense set in $L_p \Rightarrow |h| \leq 1$ on L_p ↯

Hardy

H_p

$$H_p = \left\{ f : f \text{ analytic in } |z|<1, \ \sup_{r<1} \left[\int |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right]^{1/p} < \infty \right\}$$

A few basic facts from Duren's book on H_p spaces

① Suppose $f \in H_p$. Then $\exists f^*$ defined on $T = \{|z|=1\}$

s.t.

a) $f^*(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})$ a.e.

b) $\| f^* - f_r \|_p \to 0$ as $r \to 1^-$ (where $f_r(e^{i\theta}) = f(re^{i\theta})$)

normalized Lebesgue measure on T

c) $\| f^* \|_p = \| f \|_p$

H_p norm

② If $f \in H_p$ and $|z|<1$, then $|f(z)| \le 2^{1/p} \|f\|_p (1-|z|)^{-1/p}$

<u>PROPOSITION</u>: (a) H_p^* separates the points of H_p

(b) H_p can be thought of as a closed subspace of $L_p(T, \frac{d\theta}{2\pi})$

Proof. (a) follows from ② ($f \mapsto f(z)$ cont. for each fixed z)

8/29 L_p $(0 \le p < 1)$

<u>PROPOSITION:</u> H_p is a closed subspace of $L_p(T)$

Proof. The map $f \mapsto f^*$ is a linear isometry of H_p into $L_p(T)$
$(f^*(e^{i\theta}) =$

To see that H_p is complete suppose (f_n) is Cauchy in H_p.
From

$$|f(z)| \le 2^{1/p} \|f\|_p (1-|z|)^{-1/p}$$

it follows that (f_n) is uniformly Cauchy on compact subsets of the open unit disk D. A normal family argument says that there is a function f analytic on D such that $f_n \to f$ unif on compact subsets of D. Now fix r

$$\int |f(re^{i\theta}) - f_n(re^{i\theta})|^p d\theta$$

$$\le \int |f(re^{i\theta}) - f_m(re^{i\theta})|^p d\theta + \int |f_m(re^{i\theta}) - f_n(re^{i\theta})|^p$$

$\uparrow$ may depend on r

$\le \varepsilon$ if m,n large enough

<u>PROPOSITION:</u> H_p is not locally convex

Proof (J. Roberts) Shall show $\exists$ seq $\varepsilon_n \to 0$, $\varepsilon_n > 0$, such that $z^n \in \operatorname{conv} B(0, \varepsilon_n)$.

Fact. Suppose $\mathfrak{X}^* = \{0\}$ and $\mathcal{Y}$ is a dense subspace of $\mathfrak{X}$. Then $\mathcal{Y}^* = \{0\}$.

Now suppose $f \in H_p$. We know that

$$\lim_{r \to 1^-} \|f - f_r\|_p = 0$$

where

$$f_r(z) = f(rz)$$

f_r's power series converges for $|z| < 1/r$, so f_r can be approx. by a polynomial. Hence f can be approx. in H_p norm by polynomials

Let $\mathcal{P}$ = space of all complex polynomials (dense subspace of H_p)

Consider

$$\bigcup_{n=1}^{\infty} z^{-n} \mathcal{P}$$

(i.e. all polynomials in z and $\bar{z}$ for $|z| = 1$) This latter is dense in $L_p(\mathbb{T})$ (approx L_p function by cont. function, approx that by its Fourier series. In particular, let $\varepsilon_n > 0$ be given

$$1 = \sum_{k=1}^{q_n} \alpha_k f_k \quad (\text{convex sum}) \quad \|f_k\|_p \leq \varepsilon_n$$

where $\bigcup_{n=1}^{\infty} z^{-n} \mathcal{P}$. For some fixed k_n, $f_k \in z^{-n_k} \mathcal{P}$ $\forall k \leq q_n$

$$1 = \sum_{k=1}^{q_n} \alpha_k z^{-k_n} p_k$$

$\|p_k\|_p < \varepsilon_n$

(p_k polynomial). (Polynomials in z and $\bar{z}$ are dense in L_p hence have trivial dual - so convex hull of every ball is whole space)

$$z^{k_n} = \sum_{k=1}^{q_n} \alpha_k p_k$$

Hence z^{k_n} is a convex combination of elements of H_p of norm $< \varepsilon_n$

But if $q > k_n$,

$$z^q = z^{q-k_n} z^{k_n} = \sum_{k=1}^{q_n} \alpha_k (p_k z^{q-k_n})$$

↑ H_p fn., norm $< \varepsilon_n$

▨

<u>DEFINITION:</u> $(\mathcal{X}, \|\cdot\|)$ is p-convex $(0<p<1)$

if

(a) $\|x+y\| \le K(\|x\| + \|y\|)$

(b) $\|\alpha x\| = |\alpha| \|x\|$

(c) $\|x+y\|^p \le \|x\|^p + \|y\|^p$

(Examples ℓ_p, L_p, H_p)

Fact - $\|\cdot\|$ p-convex $\Rightarrow$ $\|\cdot\|$ q-convex for $q < p$

<u>THEOREM</u>: If $(\mathfrak{X}, \|\cdot\|)$ is a separable p-convex space, then there exists a continuous linear map $T: \ell_p \xrightarrow{\text{onto}} \mathfrak{X}$

Proof. Let $S = \{x \in \mathfrak{X} : \|x\| = 1\}$. Let (x_n) be a dense sequence in S. Define $T: \ell_p \to \mathfrak{X}$ by

$$T\left(\sum \alpha_i e_i\right) = \sum \alpha_i x_i$$

Now

$$\left\| \sum_m^n \alpha_i x_i \right\|^p \leq \sum_m^n \|\alpha_i x_i\|^p \leq \sum_m^n |\alpha_i|^p$$

This says the series defining T converges and that T is continuous.

T is onto: Suppose $x \in \mathfrak{X}$, $\|x\| = 1$. Pick $x_{n_1} \in S$ s.t. $\|x - x_{n_1}\| < 1/2$

$$x = x_{n_1} + \underbrace{(x - x_{n_1})}_{r_1}$$

Pick $x_{n_2} \in S$ s.t. $\left\| \frac{r_1}{\|r_1\|} - x_{n_2} \right\| < 1/4$ $\quad (n_2 > n_1)$

$$x = x_{n_1} + \|r_1\| x_{n_2} + r_2$$

$$\|r_2\| < \frac{1}{2} \cdot 1/4$$

Continue by induction $\sum \|r_i\|^p < \infty$

$$x = \sum_{k=1}^{\infty} \|r_{n_k}\| x_{n_k}$$

9/3 L_p $(p<1)$

L_p for $0<p<1$ is a quotient of ℓ_p, i.e. $\exists\, T: \ell_p \xrightarrow{\text{onto}} L_p$. If $M = \ker T$, then ℓ_p/M is isomorphic to L_p. Hence

$$M^{\perp} = \left(\ell_p/_M\right)^{*} = L_p^{*} = \{0\}$$

So any linear functional on ℓ_p which vanishes on M is identically zero. M is called a proper closed weakly dense subspace of ℓ_p (PCWD subspace)

<u>Open problem</u>: Does every non-locally convex F space with separating dual have a PCWD subspace?

Fact: For $p<1$, ℓ_p is a quotient of H_p (via interpolating sequences)

$$H_p \xrightarrow{\text{onto}} \ell_p \xrightarrow{\text{onto}} L_p$$

So same argument produces a PCWD subspace of H_p

Let $M = \ker T$ in ℓ_p as above. Pick $x_0 \in \ell_p \setminus M$. Then $\mathrm{sp}\{x_0, M\} = \mathbb{R}x_0 \oplus M$ which is closed. There is a continuous linear functional on $\mathbb{R}x_0 \oplus M$ which is non-zero and vanishes on M. That linear functional has no continuous linear extension to ℓ_p, i.e. $M \oplus \mathbb{R}x_0$ fails the Hahn-Banach extension property in ℓ_p (HBEP)

<u>DEFINITION</u>: A closed subspace M of E has HBEP in E

if every continuous linear functional on M extends to a continuous linear functional on E.

<u>THEOREM</u> (Kalton) If E is an F-space and if every closed subspace of E has HBEP in E, then E is locally convex.

<u>THEOREM</u> Every closed infinite dimensional subspace of ℓ_p contains an infinite dimensional subspace isomorphic to ℓ_p

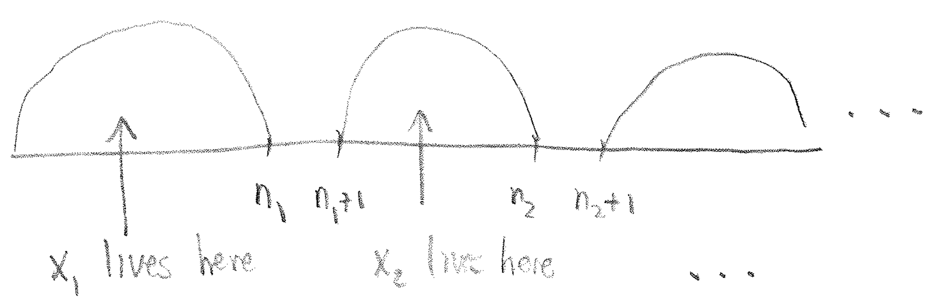
Proof. (Gliding hump still works) Given $E \subset \ell_p$, E inf. dim. Pick $x^1 \in E$, $\|x^1\| = 1$. Pick n_1 s.t.

$$\sum_{k=n_1+1}^{\infty} |x^1_k|^p < \varepsilon$$

$\dim E = \infty \implies \exists x_2 \in E$ s.t. $x^2_i = 0$ for $i = 1, 2, \dots, n_1$ and $\|x^2\| = 1$. Pick $n_2 > n_1$ s.t.

$$\sum_{k=n_2+1}^{\infty} |x^2_k|^p < \varepsilon$$

Continue by induction



$$\Big\| \sum_{k=1}^{j} \alpha_k x^k \Big\|_p^p \le \sum_{k=1}^{j} |\alpha_k|^p \qquad (\text{p-convexity})$$

$$\Big\| \sum_{k=1}^{j} \alpha_k x^k \Big\|_p^p \ge \overset{①}{|\alpha_1|^p \sum_{i=1}^{n_1} |x_i^1|^p} + \overset{②}{\sum_{i=n_1+1}^{n_2} |\alpha_1 x_i^1 + \alpha_2 x_i^2|^p}$$

$$+ \dots + \sum_{i=n_{j-1}}^{n_j} |\alpha_1 x_i^1 + \dots + \alpha_j x_i^j|^p$$

$$① \ge |\alpha_1|^p \Big(\sum_{i=1}^{\infty} |x_i|^p - \sum_{i=n_1+1}^{\infty} |x_i^1|^p \Big) \ge |\alpha_1|^p (1-\varepsilon)$$

$$② \ge \sum_{i=n_1+1}^{n_2} |\alpha_2|^p |x_i^2|^p - \sum_{i=n_1+1}^{n_2} |\alpha_1|^p |x_i^1|^p$$

$$\ge |\alpha_2|^p (1-\varepsilon) - \sum_{i=n_1+1}^{n_2} |\alpha_1|^p |x_i^1|^p$$

$$③ \ge |\alpha_3|^p (1-\varepsilon) - \underbrace{\sum_{i=n_2+1}^{n_3} |\alpha_1|^p |x_i^1|^p}_{\text{add up } \le |\alpha_1|^p \varepsilon} - \underbrace{\sum_{i=n_2+1}^{n_3} |\alpha_2|^p |x_i^2|^p}_{\text{add up } \le |\alpha_2|^p \varepsilon}$$

$$\therefore \Big\| \sum_{i=1}^{j} \alpha_i x^i \Big\|_p^p \ge \Big(\sum_{i=1}^{j} |\alpha_i|^p \Big) (1 - \varepsilon - \varepsilon) \qquad \blacksquare$$

§2 L_0

<u>LEMMA</u> Let $T : L_0 \longrightarrow L_0$ be continuous. Given $\varepsilon > 0$ there is a $\delta > 0$ s.t.

$$m(\operatorname{supp} f) < \delta \Rightarrow m(\operatorname{supp} Tf) \leq \varepsilon$$

$(\operatorname{supp} f = \{x : f(x) \neq 0\})$

Proof. Given ε, choose δ from the continuity of T. Suppose $m(\operatorname{supp} f) < \delta$. Then for any $n \in \mathbb{N}$

$$\int \frac{|nf|}{1+|nf|} < \delta$$

$$\Rightarrow \int \frac{n|Tf|}{1+n|Tf|} < \varepsilon$$

Let $n \to \infty$ $\quad \dfrac{n|Tf|}{1+n|Tf|} \longrightarrow \chi_{\operatorname{supp}|Tf|}$. Hence by DCT

$$m(\operatorname{supp} Tf) < \varepsilon$$

9/5 L_p $(0 \le p < 1)$

<u>LEMMA:</u> Suppose $f_1, \dots, f_n \in L_0$. Then $\exists\, g = \sum_{i=1}^{n} \alpha_i f_i$ s.t.

$$\operatorname{supp} g = \bigcup_{i=1}^{n} \operatorname{supp} f_i \quad \text{(up to sets of measure 0)}$$

Proof. (Enough to show for $n=2$) Given $f_1, f_2 \in L_0$, let $A_i = \operatorname{supp} f_i$. Let $A = A_1 \cup A_2$. For $t \in \mathbb{R}$, let

$$A_t = \{x \in A : f_1(x) + t f_2(x) = 0\}$$

Claim: If $t_1 \neq t_2$, then $A_{t_1} \cap A_{t_2} = \emptyset$

The A_t's are measurable sets and pairwise disjoint. So $\exists\, t_1$ s.t. $m(A_{t_1}) = 0$. Then

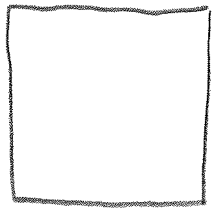
$$x \in A \setminus A_{t_1} \Rightarrow f_1(x) + t_1 f_2(x) \neq 0$$

$$\Rightarrow x \in \operatorname{supp}(f_1 + t f_2)$$

$$\therefore \operatorname{supp} f_1 \cup \operatorname{supp} f_2 \subset \operatorname{supp}(f_1 + t_1 f_2) \quad \text{"a.e."}$$

Always have $\operatorname{supp}(f_1 + t_1 f_2) \subset \operatorname{supp} f_1 \cup \operatorname{supp} f_2$

$\blacksquare$



m^2 = 2-dim Lebesgue measure on $I \times I$

$$(Pf)(x,y) := \int_0^1 f(x,t)\, dm(t)$$

is usual conditional expectation operator on $L_p(I \times I)$ $p \geq 1$

$$\|Pf\|_p \leq \|f\|_p \qquad p \geq 1$$

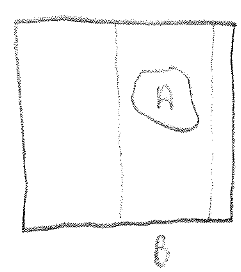
$$P^2 = P$$

Range of P = all functions constant on vertical lines

THEOREM (Berg, Porta, Peck) There is no continuous projection of $L_0(I \times I)$ onto functions constant on vertical lines

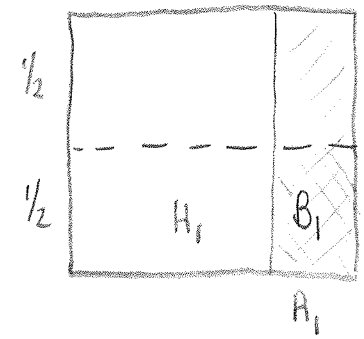
Will prove $\exists$ no positive continuous projection of $L_0(I \times I)$ onto functions constant on vertical lines.

Proof. Suppose $0 \leq f \leq \chi_A$, where $A \subset B \times I$
Then $0 \leq Pf \leq P\chi_A \leq P\chi_{B \times I} = \chi_{B \times I}$



Claim: $\forall n \in \mathbb{N}$ $\exists$ function f s.t.
$m^2(\operatorname{supp} f) \leq 1/n$ yet $m^2(\operatorname{supp} Pf) = 1$
which will contradict continuity of P

Take $n=2$

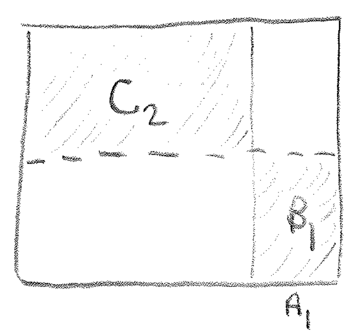


$P\chi_{H_1} =: \mathfrak{F}_1$

a) $\text{supp } \mathfrak{F}_1 = A_1 \times I$

b) $B_1 = H_1 \cap (A_1 \times I)$ $\quad$ $\text{supp } P\chi_{B_1} = A_1 \times I$

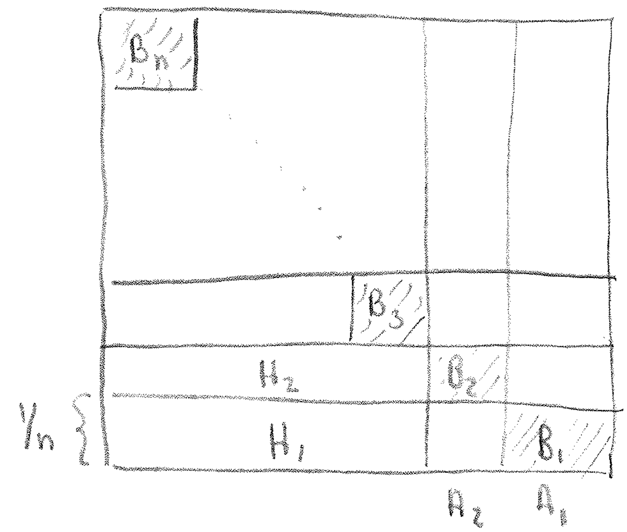
$\subset$ by preceeding remark
$\supset$ by considering $\chi_{H_1 \setminus B_1}$
$\text{supp}(P\chi_{H_1 \setminus B_1}) \cap A_1 \times I = \emptyset$



$\text{supp } P\chi_{C_2} = A_1^c \times I$

Note: $m^2(C_2 \cup B_1) = 1/2$ but $m^2(\text{supp } P\chi_{C_2 \cup B_1}) = 1$

For general n



$$\operatorname{supp} P\chi_{H_1} = A_1 \times I = \operatorname{supp} P\chi_{B_1}$$

$$\operatorname{supp} P\chi_{H_2} = A_2 \times I = \operatorname{supp} P\chi_{B_2}$$

$$\vdots$$

$$f = \sum \chi_{B_i}$$

$$\therefore \; m^2(\operatorname{supp} f) = \sum 1/n \, m(A_i) = 1/n$$

$$m^2(\operatorname{supp} Pf) = \sum m^2(A_i \times I) = 1$$

$\blacksquare$

Lifting Theorems in $L_0[0,1]$

Suppose $f \in L_0$, $f \neq 0$. Is $L_0/[f]$ linearly homeomorphic (isomorphic) to L_0? NO

span$(\{f\})$

Lifting Theorem: Given $T : L_0 \longrightarrow L_0/F$ where F is finite dimensional, there exists $\tilde{T} : L_0 \longrightarrow L_0$

$$\begin{tikzcd}
 & L_0 \arrow[d, "\pi"] \\
L_0 \arrow[ur, dashed, "\tilde{T}"] \arrow[r, "T"'] & L_0/F
\end{tikzcd}$$

such that $T = \pi \tilde{T}$, where π is canonical quotient map of L_0 onto L_0/F.

9/8 L_p $(0 \leq p < 1)$

<u>LIFTING THEOREM:</u> $T : L_0 \longrightarrow L_0/F$, F finite dim. subspace.
Then there is a unique linear operator $\tilde{T} : L_0 \longrightarrow L_0$ s.t. $T = \pi \circ \tilde{T}$

$$\begin{tikzcd} & L_0 \arrow[d, "\pi"] \\ L_0 \arrow[ur, "\tilde{T}"] \arrow[r, "T"'] & L_0/F \end{tikzcd}$$

Proof. Uniqueness is easy. Suppose $\pi T_1 = T$ also. Then $\pi(\tilde{T} - T_1) = 0$ and so $(\tilde{T} - T_1) : L_0 \longrightarrow F$. But F^* separates points so if $\tilde{T} - T_1 \neq 0$, there would exist non-zero continuous functionals on L_0. Since $L_0^* = \{0\}$, $\tilde{T} = T_1$

Is L_0 isomorphic to L_0/F where F is a non-zero finite dim subspace?
Suppose $T : L_0 \longrightarrow L_0/F$ is an isomorphism. Lift T to $\tilde{T} : L_0 \longrightarrow L_0$

① $\tilde{T}$ is 1-1
② $\tilde{T}$ is an isomorphism
③ $\tilde{T}(L_0)$ is a closed subspace of codimension = dim F. This is impossible in a space with no continuous functionals (write $L_0 = Z \oplus F$

$\uparrow$ dim $< \infty$

$$x^*(g) = x^*(z + f) = \underset{\text{on } F}{\text{linear functional}}\,(f)$$

(Resume proof) The topology on F can be given by a norm $\|\cdot\|$ since $\dim F < \infty$.

Step 1: $\exists c > 0$ such that if $x \in F$, $(x \neq 0)$, then $m(\operatorname{supp} x) > c$

To see this, suppose not. Then $\exists (x_n) \subset F$ s.t. $m(\operatorname{supp} x_n) \to 0$. Let $y_n = x_n / \|x_n\|$. Then $m(\operatorname{supp} y_n) \to 0$ and so $y_n \to 0$ (L_0 norm) $\Rightarrow y_n \to 0$ in F ($\|\cdot\|$ norm) $\lightning$

Step 2: $\exists \delta > 0$ s.t. if $m(\operatorname{supp} f) < \delta$, then $\exists!\, h(f) \in Tf \in L_0/F$ s.t. $m(\operatorname{supp} h(f)) \leq c/3$

(Proof later)

For proving the theorem, let $(A_i)_{i=1}^n$ be a partition of $[0,1]$ with $m(A_i) < \delta$ $1 \leq i \leq n$. For $f \in L_0$, define

$$\tilde{T} f := \sum_{i=1}^n h(f \chi_{A_i})$$

Then $T = \pi \tilde{T}$

Must check $\tilde{T}$ is linear and continuous

a) homogeneity of h: $h(\alpha f) = \alpha h(f)$ by uniqueness since both lie in $T(f)$ and have support $\leq c/3$ in measure

b) additivity of h: $h(f+g) - h(f) - h(g) \in T(0) = F$ and $m(h(f+g) - h(f) - h(g)) \leq c/3 + c/3 + c/3 = c$. By step 1, this must be 0 (for (b) assuming $m(\mathrm{supp}(f+g)) < \delta$, $m(\mathrm{supp} f) < \delta$, $m(\mathrm{supp}\, g) < \delta$)

Hence $\tilde{T}$ is linear

Continuity: Suppose f_n is supported in A, $m(A) < \delta$. Want to show that if $f_n \to 0$ in measure then $h(f_n) \to 0$ in measure. We know $Tf_n \to 0$, so $\exists\ (w_n) \subset F$ s.t.

$$(*) \qquad h(f_n) + w_n \longrightarrow 0 \text{ in measure}$$

Claim: $w_n \to 0$. For if not, by passing to a subsequence if necessary, $\|w_n\| \geq \varepsilon > 0$

$$\frac{w_n}{\|w_n\|} \in F$$

By compactness, WLOG $w_n/\|w_n\| \longrightarrow w \in F$

$$\therefore \quad \frac{h(f_n)}{\|w_n\|} \longrightarrow -w \qquad \text{by } (*)$$

$$\uparrow \qquad\qquad \uparrow$$

$m(\mathrm{supp}) \leq c/3$ $\qquad$ $m(\mathrm{supp}) > c$ by step 1

9/10 Lp

Proof of step 2: Uniqueness - suppose $h(f)$ and q satisfies conclusion. Then $h(f) - q \in F$ and $\text{supp}\,(h(f)-q)$ has measure $\leq 2c/3 < c$, so $h(f) - q = 0$.

By continuity of T $\exists\, \delta > 0$ s.t.

$$\|f\|_0 < \delta \implies \underbrace{|||\, Tf \,|||}_{\text{quotient norm}} < c/3$$

Now suppose $m(\text{supp}\, f) < \delta$. Then for any $n \in \mathbb{N}$, $\|nf\|_0 < \delta$ and so $|||\, T(nf)\,||| < c/3$. Fix $z \in Tf$. $\exists\, w_n \in F$ s.t.

$$(*) \qquad \int \frac{|nz + w_n|}{1 + |nz + w_n|}\, dm < c/3$$

Case 1: (w_n/n) has a bounded subsequence $\left(\frac{\|w_n\|}{n} < M \right)$ WLOG

Passing to a subseq. if necessary, we may assume $w_n/n \longrightarrow w$ in the norm on F and a.e. In $(*)$ we obtain

$$\int \underbrace{\frac{|z + w_n/n|}{1/n + |z + w_n/n|}}_{\text{this converges to } 1 \text{ on } \text{supp}(z+w)}\, dm < c/3$$

$\therefore\ m(\text{supp}\,(z+w)) \leq c/3$

Case 2: $\left\| \frac{w_n}{n} \right\| \to \infty$. Hence $\|w_n\| \to \infty$. Now in (*) we have

$$\int \frac{\left| \frac{n}{\|w_n\|} z + \frac{w_n}{\|w_n\|} \right|}{1/\|w_n\| + \left| \frac{n}{\|w_n\|} z + \frac{w_n}{\|w_n\|} \right|} \, dm < c/3$$

Pass to subseq to get $w_n / \|w_n\| \to w \in F$ in F-norm and a.e. Now the integrand tends a.e. to 1 on supp w. Hence

$$m(\operatorname{supp} w) \leq c/3$$

$\blacksquare$

<u>DEFINITION</u>: An F-space $\mathfrak{X}$ has L_0-structure if for every $\varepsilon > 0$,

$$\mathfrak{X} = \bigoplus_{i=1}^{n} \mathfrak{X}_i$$

($\mathfrak{X}_i$ closed subspace) s.t. $\operatorname{diam}(\mathfrak{X}_i) < \varepsilon$.

L_0 has L_0-structure: Given $\varepsilon > 0$ partition $[0,1]$ into intervals of length $< \varepsilon$. $[0,1] = \bigcup_{i=1}^{n} A_i$ $\quad \mathfrak{X}_i = \{ f \chi_{A_i} : f \in L_0 \}$

For any F-space $\mathcal{Y}$, $\mathfrak{X} = L_0(\mathcal{Y})$ = all $\mathcal{Y}$-valued measurable functions. Then $L_0(\mathcal{Y})$ has L_0-structure

$$L_0(Y) = \bigoplus L_0(A_i, Y)$$

$\uparrow$ as before

(Question: What do spaces with L_0 structure look like?)

__DEFINITION:__ Suppose $\mathcal{X}$ is an F-space. Define

$$\sigma(x) = \sup_{r \in \mathbb{R}} |rx|$$

__DEFINITION:__ An F-space $\mathcal{X}$ is locally bounded if its topology has a bounded neighborhood of 0.

(U bdd nbhd $\Rightarrow$ if V nbhd of then $U \subset nV$ for some n)

In this case topology can be given by a quasi-norm $\|\cdot\|$

a) $\|x+y\| \leq K(\|x\| + \|y\|)$
b) $\|\alpha x\| = |\alpha| \|x\|$

May assume U is symmetric w.r.t. 0, then take $\|\cdot\|$ = guage functional of U.

<u>THEOREM</u> (Generalized Lifting theorem) Let $\mathfrak{X}$ be an F-space with L_0-structure. Let $\mathcal{Y}$ be an F-space and let B be a closed locally bounded subspace of $\mathcal{Y}$. Let $T : \mathfrak{X} \longrightarrow \mathcal{Y}/B$. Then there is a unique lifting of T to $\tilde{T} : \mathfrak{X} \longrightarrow \mathcal{Y}$

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "T"] \arrow[dr, dashed, "\tilde{T}"'] & \mathcal{Y}/B \\ & \mathcal{Y} \arrow[u, "\pi"'] \end{tikzcd}$$

Proof. uniqueness of $\tilde{T}$: If $T = \pi T_1 = \pi \tilde{T}$, then $\pi(\tilde{T} - T_1) = 0$, so $\tilde{T} - T_1 : \mathfrak{X} \longrightarrow B$

L_0 str. loc. bdd (homogen norm) $\therefore \tilde{T} - T_1 = 0$

Key lemma: Choose $\delta > 0$ s.t. $\{ b \in B : |b| \leq \delta \}$ is a bounded nbhd of 0. Then if $z \in \mathcal{Y}/B$ and $\sigma(z) \leq \delta/3$, then $\exists$ unique $y \in \mathcal{Y}$ s.t. $\pi y = z$ and $\sigma(y) \leq \delta/3$

uniqueness of y: If $\pi y_1 = z$ and $\sigma(y_1) \leq \delta/3$, then $\pi(y_1 - y) = 0$ and $\sigma(y_1 - y) \leq 2\delta/3 < \delta$, i.e.

$$|n(y_1 - y)| < \delta \quad \forall n$$

But $y_1 - y \in B$ so this last inequality is impossible since we have a bounded nbhd of 0. Hence $y_1 - y = 0$

9/12 L_p

Proof of lifting theorem

Step 1: Take $\delta > 0$ s.t. $\{b \in B : |b| \le \delta\}$ is bounded. Then given $z \in Y/B$ with $\sigma(z) \le \delta/3$, $\exists$ unique $x \in Y$ with $\pi x = z$ so that $\sigma(x) = \sigma(z)$

(proof later)

Step 2: Let H be a linear subspace of Y/B with $\operatorname{diam}(H) \le \delta/6$. Then $\exists$ a continuous linear operator $V : H \to Y$ s.t. $\pi V = I_H$

Proof. Suppose $z \in H$. Then $\sigma(z) \le \delta/6$. Define $V(z)$ to be the unique y from step 1 with $\pi y = z$, $\sigma(y) = \sigma(z)$. By uniqueness of y, V is linear

For continuity of V, suppose $z_n \to 0$ $(z_n \in H)$

Choose $x_n \in Y$ such that $\pi x_n = z_n$ and $|x_n| \le 2|z_n|$. Hence $x_n \to 0$. Shall show $x_n - Vz_n \to 0$ (so $Vz_n \to 0$)

Now $x_n - Vz_n \in B$. Suppose the sequence does not converge to 0. Pass to a subsequence to get $|x_n - Vz_n| \ge \gamma > 0$. For sufficiently large α

$$|\alpha(x_n - Vz_n)| \ge \delta$$

(since B is locally bdd). Then

$$|\alpha V z_n| \geq \delta - |\alpha x_n| \geq \delta - \delta/6 = 5\delta/6 \text{ for suff. large } n$$

But $\sigma(Vz_n) \leq \delta/3$ so this is impossible. $[\sigma(\alpha p) = \sigma(p)]$

Proof of th^m: Take $\varepsilon = \delta/6$. Choose $\gamma > 0$ s.t. if $|x| < \gamma$, then $|Tx| < \delta/6$. Now X has L_0-structure, so

$$\mathfrak{X} = \bigoplus_{i=1}^{n} \mathfrak{X}_i \quad (\operatorname{diam} \mathfrak{X}_i < \gamma)$$

$\sigma(x) \leq \gamma$ for $x \in \mathfrak{X}_i$

$$\therefore \quad T\mathfrak{X} = \sum_{i=1}^{n} T\mathfrak{X}_i$$

and if $z \in T\mathfrak{X}_i$, then $\sigma(z) \leq \delta/6$

Let $H_i = T\mathfrak{X}_i$, $i = 1, \ldots, n$. Apply step 2 to each H_i. Then $\exists\ V_i : H_i \to Y$ where V_i is cont., linear and $\pi V_i = I_{H_i}$. The desired lifting is

$$\tilde{T}\left(\sum_{i=1}^{n} x_i\right) := \sum_{i=1}^{n} V_i(Tx_i)$$

Proof of step 1: For each integer n, pick x_n s.t. $\pi x_n = z$ and $|nx_n| \leq (1 + 1/n)|nz|$

Claim - (x_n) is a Cauchy sequence

To see this let $v_n = x_n - x_1 \in B$.

For $2 \le m < n$

$$|mu_n - mu_m| = |m(x_n - x_m)| \le |mx_n| + |mx_m|$$

$$\le |nx_n| + |mx_m| \quad (\text{F-norm increasing})$$

$$\le (1 + 1/n)\,\delta/3 + (1 + 1/m)\,\delta/3$$

$$= (2 + 1/n + 1/m)\,\delta/3 \le \delta$$

Since $u_n - u_m$ lies in a bdd nbhd of 0, the above inequality shows that $\lim_{n,m\to\infty} |u_n - u_m| = 0$, i.e. (u_n) is Cauchy. Hence

(x_n) is Cauchy, so $x_n \to x$

Now fix $n \in \mathbb{N}$

$$|nx| \le |nx_m| + |n(x_m - x)|$$

$$\le (1 + 1/m)\sigma(z) + \varepsilon$$

for sufficiently large m. Hence $|nx| \le \sigma(z)\ \forall n \Rightarrow \sigma(x) \le \sigma(z)$

Already have $\sigma(z) \le \sigma(x)$.

9/15 L_p

Suppose $T: \mathcal{Y} \xrightarrow{\text{onto}} \mathfrak{X}$ with $\dim(\ker T) = 1$.

$\mathcal{Y}$: F-space; $\mathfrak{X}$: L_0-structure

can replace with $\ker T$ locally bdd

Then $\ker T$ is complemented in $\mathcal{Y}$

Proof. Define $\hat{T}$ on $\mathcal{Y}/\ker T \cong \mathfrak{X}$ by $\hat{T}(y + \ker T) = Ty$
$\hat{T}$ is 1-1, onto, bicontinuous.

$$\hat{T}^{-1} : \mathfrak{X} \longrightarrow \mathcal{Y}/\ker T$$

Now use lifting theorem to obtain $V: \mathfrak{X} \to \mathcal{Y}$ such that $\pi V = \hat{T}^{-1}$. Let $x \in \mathfrak{X}$. Then

$$x = \hat{T}(\hat{T}^{-1}x) = \hat{T}(\pi Vx) = \hat{T}(Vx + \ker T) = TVx$$

Hence $TV = I_{\mathfrak{X}}$. From this it's easy to see that $\ker T$ is complemented (VT is a projection)

DEFINITION: $\mathfrak{X}$ is a K-space if whenever

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{Y} \xrightarrow{T} \mathfrak{X} \longrightarrow 0$$

then $\exists\ V : \mathfrak{X} \to \mathcal{Y}$ s.t. $TV = I_{\mathfrak{X}}$

L_0, L_p, ℓ_p $(p<1)$ K-spaces ; ℓ_1 is not a K-space

Operators on L_0

(Work of Kwapien)

$$L_0(I,\mathcal{B},m) \xrightarrow{T} L_0(I,\mathcal{B},m)$$

Example: $Tf(x) := \varphi(x) f(\Phi x)$ where

$\varphi : I \longrightarrow \mathbb{R}$ is measurable
$\Phi : I \longrightarrow I$ is measurable

Suppose $m(A)=0$. We want $T(\chi_A)=0$ a.e. This means

$$T\chi_A = \chi_{\Phi^{-1}(A)} = 0 \text{ a.e.}$$

i.e. $m(\Phi^{-1}(A))=0$. If ν on $[0,1]$ is given by $\nu(A)=m(\Phi^{-1}(A))$
Then $\nu \ll m$, i.e.

(*) given $\varepsilon>0$ $\exists\, \delta>0$ s.t. $m(A)<\delta \Rightarrow m(\Phi^{-1}(A))<\varepsilon$

Suppose Φ satisfies (*). Then $f \mapsto f\circ\Phi$ is a continuous linear operator on L_0. Have just seen cosets are preserved
To see continuity, suppose $m\{|f|>\delta\}\le\delta$. Then

$$m\{|Tf|>\delta\} \le m(\Phi^{-1}\{|f|>\delta\}) < \varepsilon$$

Now $f \mapsto \varphi(f\circ\Phi)$ is also continuous ($f_n \xrightarrow{m} 0,\ g_n \xrightarrow{m} 0 \Rightarrow f_n g_n \xrightarrow{m} 0$)

THEOREM (Kwapien) Let (φ_i) be a sequence of measurable functions from I to $\mathbb{R}$. Let (Φ_i) be a sequence of measurable functions from I to I s.t.

① for almost all t $\{n \in \mathbb{N} : \varphi_n(t) \neq 0\}$ is finite

② $m(A) = 0 \Rightarrow m(\Phi_n^{-1}(A)) = 0 \quad \forall n$

Then $Tf(t) := \sum_n \varphi_n(t)(f \circ \Phi_n(t))$ defines a continuous linear operator on L_0.

Conversely, every continuous linear operator on L_0 has this form.

Proof. $(\Rightarrow)$ Note $f \longrightarrow \sum_{n=1}^{k} \varphi_n(f \circ \Phi_n)$ is a continuous linear operator. By ①, $T_k f \sum_{i=1}^{k} \varphi_i(f\Phi_i) \longrightarrow \sum_{i=1}^{\infty} \varphi_i(f\Phi_i)$ a.e. and thus in measure. Hence $T_k f \longrightarrow Tf \ \forall f$, so by uniform boundedness, T is continuous.

$(\Leftarrow)$ Notation: $\Delta_k^n = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$ dyadic interval

$$W_k^n = T\chi_{\Delta_k^n}$$

$$A_k^n = \operatorname{supp} W_k^n \qquad (1 \le k \le 2^n)$$

$$\chi_k^n = \chi_{A_k^n}$$

Note that $W^n_k = W^{n+1}_{2k-1} + W^{n+1}_{2k}$, so $\chi^n_k \le \chi^{n+1}_{2k-1} + \chi^{n+1}_{2k}$

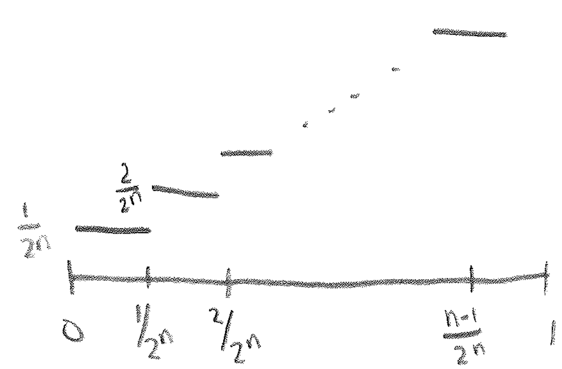
For each n and i, define

$$\Phi^n_i(t) = \frac{\min\left\{k : \sum_{j=1}^{k} \chi^n_j(t) \ge i\right\}}{2^n}, \text{ if is not } \emptyset$$

$$= 1, \text{ otherwise}$$

(Consider $Tf(x) = f(x)$.

Φ^n_1



$\Phi^n_2 = 1$)

Define $\varphi^n_i(t) = W^n_k$ if $\Phi^n_i(t) = k/2^n$

$= 0$, otherwise

9/17 L_p

(Proof continued)

$\forall \varepsilon > 0 \ \exists M$ s.t. $m\{|h_n| > M\} < \varepsilon \ \forall n$

We will prove $\left(\sum_{k=1}^{2^n} \chi_k^n \right)$ is L_0-bounded. Note

$$\chi_k^n \leq \chi_{2k-1}^{n+1} + \chi_{2k}^{n+1} \Rightarrow \text{seq is increasing}$$

LEMMA: An increasing, L_0-bounded sequence (h_n) is pointwise bounded, i.e. $h = \sup h_n$ exists a.e.

Proof that $\sum_{k=1}^{2^n} \chi_k^n$ is L_0-bounded. Let $\varepsilon > 0$. Chose $\delta > 0$

s.t. if $m(|\xi| > \delta) < \delta$, then $m(|T\xi| > \varepsilon) < \varepsilon$. Suppose K is an integer with $K \leq \delta 2^n$. Let $c_{i_1}, \ldots, c_{i_K}$ be any scaler $(i_K < 2^n)$ Then

$$m\left(|\xi = \sum_{j=1}^{k} c_{i_j} \chi_{\Delta_{i_j}^n}| > \delta \right) < \delta$$

$$\Rightarrow m\left(|\sum_{j=1}^{k} c_{i_j} w_{i_j}^n| > \varepsilon \right) < \varepsilon$$

The c_{i_j} can be arbitrary, so

$$m\left(|\sum_{j=1}^{k} c_{i_j} w_{i_j}^n| > 0 \right) < \varepsilon$$

If $g_1, \dots, g_i$ are measurable functions, there exist linear combination

$$\sum c_i g_i$$

s.t. $\operatorname{supp}\left(\sum c_i g_i\right) = \bigcup \operatorname{supp} g_i$. By an early lemma there exists scalers $c_{i_1}, \cdots c_{i_k}$ s.t

$$\operatorname{supp} \left| \sum c_{i_j} w^n_{i_j} \right| = \bigcup A^n_{i_j}$$

So given $\varepsilon > 0$ $\exists\, \delta > 0$ s.t. if $k \leq \delta \cdot 2^n$, then

$$m\left(\bigcup_{j=1}^{k} A^n_{i_j} \right) \leq \varepsilon$$

LEMMA: Let $\varepsilon, \delta > 0$, m integer, M integer, $A_1, \dots, A_m$ measurable sets s.t. if $k \leq \delta m$, then

$$m\left(\bigcup_{j=1}^{k} A_{i_j} \right) < \varepsilon$$

for all $i_1, \dots, i_k \leq m$. Then

$$m\left(\left| \sum_{j=1}^{m} \chi_{A_j} \right| \geq M \right) \leq \frac{\varepsilon + \frac{2-\delta}{m\delta}}{1 - \left(1 - \frac{\delta}{2}\right)^M}$$

Proof of lemma: Let $\Gamma_1, \dots, \Gamma_m$ be independent random variables on a probability space (Ω, Σ, P) having distribution

$$P(\Gamma_i = 1) = \delta/2 \qquad \text{(binomial)}$$

$$P(\Gamma_i = 0) = 1 - \delta/2$$

Then

$$m \otimes P \left\{ (t,\omega) : \sum_{i=1}^{m} \chi_{A_i}(t)\, \Gamma_i(\omega) \geq 1 \right\} = ?$$

First look at

$$P\left\{ \omega : \sum_{i=1}^{m} \Gamma_i(\omega) \geq m\delta \right\}$$

$$= P\left(\omega : \sum_{i=1}^{m} (\Gamma_i - E\Gamma_i) \geq m\delta/2 \right) \qquad (E\Gamma_i = \delta/2)$$

Variance $\displaystyle\sum_{i=1}^{m} \Gamma_i = m \frac{\delta}{2}\left(1 - \frac{\delta}{2}\right) = \frac{m\delta}{2}\,\frac{2-\delta}{2}$

$$\therefore \; P\left(\sum (\Gamma_i - E\Gamma_i) \geq m\delta/2 \right) \leq \frac{\frac{m\delta}{2}\,\frac{2-\delta}{2}}{m^2 \delta^2/4} = \frac{2-\delta}{m\delta}$$

(Chebychev inequality)

$$\therefore \quad P(\overbrace{\textstyle\sum_i \Gamma_i \geq m\delta}^{B}) \leq \frac{2-\delta}{m\delta}$$

Fix t. Then $\omega \in B$ or $\omega \in B^c$.

If $\sum \Gamma_i(\omega) \geq m\delta$, then $P\left(\sum_i \chi_{A_i}(t)\Gamma_i(\omega) \geq 1\right) \leq \dfrac{2-\delta}{m\delta}$

If $\sum_i \Gamma_i(\omega) < m\delta$, then $m\left(\bigcup_{j=1}^{k} A_{i_j}\right) < \varepsilon \quad (k \leq m\delta)$

$$\therefore \quad m \otimes P\left\{(t,\omega) : \sum_{i=1}^{m} \chi_{A_i}(t)\Gamma_i(\omega) \geq 1\right\} \leq \varepsilon + \frac{2-\delta}{m\delta}$$

Suppose $\exists t$ s.t. $\sum_i \chi_{A_i}(t) \geq M$. Want to look at

$$\left\{(t,\omega) : \sum_i \chi_{A_i}(t)\Gamma_i(\omega) \geq 1\right\}$$

Suppose $\sum \chi_{A_i}(t) = M$

9/19 Lp

(Proof continued)

Suppose $\sum_{i=1}^{m} \chi_{A_i}(t) = M$. Then

$$P\left(\omega : \sum_{i=1}^{m} \chi_{A_i}(t)\, r_i(\omega) \geq 1\right) = P\Big(\omega : r_i(\omega) = 0 \ \forall i \text{ s.t. } \underbrace{\chi_{A_i}(t) = 1}_{\text{this happens } M \text{ times}}\Big)$$

$$= 1 - \left(1 - \frac{\delta}{2}\right)^M$$

If $\sum \chi_{A_i}(t) \geq M$, then

$$P\left(\omega : \sum_{i=1}^{m} \chi_{A_i}(t)\, r_i(\omega) \geq 1\right) \geq 1 - \left(1 - \frac{\delta}{2}\right)^M$$

By Fubini's theorem

$$\left(1 - \left(1 - \frac{\delta}{2}\right)^M\right) m\left(\sum_{i=1}^{m} \chi_{A_i}(t) \geq M\right) \leq \varepsilon + \frac{2-\delta}{m\delta}$$

$$\therefore\ m\left(\sum_{i=1}^{m} \chi_{A_i}(t) \geq M\right) \leq \frac{\varepsilon + \frac{2-\delta}{m\delta}}{1 - (1 - \delta/2)^M}$$

This can be made small

Recall $\Phi_i^n(t) := \frac{1}{2^n} \min\{k : \sum_{j=1}^{k} \chi_{A_j^n}(t) \geq i\}$

if there is such a k ; and $= 1$ otherwise

$$\varphi_i^n(t) = \begin{cases} w_k^n(t) & \text{if } \Phi_i^n(t) = k/2^n \\ 0 & \text{if } \Phi_i^n(t) = 1 \end{cases}$$

Claim: $\Phi_i^n(t)$ converges (in n) a.e. in t

Suppose $\Phi_i^n(t) = k/2^n$. Assume $n = n(t)$ is large enough so that $g_n(t) = g_{n+1}(t)$, $n > n(t)$. Then

$$\Phi_i^{n+1}(t) = \frac{2k}{2^{n+1}} \quad \text{or} \quad \frac{2k-1}{2^{n+1}}$$

Hence

$$\Phi_i^{n+1}(t) - \Phi_i^n(t) \leq \frac{1}{2^{n+1}}$$

So let $\Phi_i(t) = \lim_{n \to \infty} \Phi_i^n(t)$

Suppose $\varphi_i^n(t) = w_k^n(t)$

$$w_k^n(t) = w_{2k-1}^{n+1}(t) + w_{2k}^{n+1}(t)$$

only one can be non-zero if $w_k^n(t) \neq 0$

Hence $\varphi_i^n(t) = w_k^n(t) = \varphi_i^{n+1}(t)$ (eventually constant). Let

$$\varphi_i(t) = \lim_{n \to \infty} \varphi_i^n(t)$$

Claim: For each t only finitely many of the values $\varphi_i(t)$ are non-negative

Suppose not. Then given M, $\varphi_{i_1}, \ldots, \varphi_{i_M}(t)$ are all non-zero (for some t)

$$\varphi_i(t) = w_k^n(t) \quad \text{if} \quad \Phi_i^n(t) = k/2^n$$

for suff. large n. Then at least M of the functions $\chi_j^n(t)$ are non-zero. Hence

$$g_n(t) = \sum_{k=1}^{2^n} \chi_j^n(t) \geq M$$

(since $g_n(t)$ is bdd)

Claim: For each i, if $m(A) = 0$, then

$$m(\Phi_i^{-1}(A)) = 0$$

(if true, define $\tilde{T}f = \sum_{i=1}^{\infty} \varphi_i(t) f(\Phi_i(t))$)

Suppose given ε. $\delta > 0$ will be chosen later. Then

$\exists$ open U s.t. $A \subset U$ and $m(U) < \delta$. Suppose $\Phi_i(t) \in U$

Then for n very large, $\Phi_i^n(t) = k/2^n$ so may assume

$\Delta_n^k \subset U$

9/22 L_p

For each i, $m(A) = 0 \Rightarrow m\,\Phi_i^{-1}(A) = 0$

Let $\varepsilon > 0$ and choose $\delta > 0$ s.t. $|f|_0 < \delta \Rightarrow |Tf|_0 < \varepsilon$. (*)
If $m(A) = 0$, then $\exists$ open $U \subset A$ s.t. $m(U) < \delta$. Suppose $\Phi_i(t) \in U$. Then for some n, k $\Delta_k^n \subset U$. Suppose $\varphi_i(t) \neq 0$
Can take n so large that

$$\varphi_i(t) = W_k^n(t) \neq 0$$

Then

$$\Phi_i^{-1}(U) \cap \{\varphi_i(t) \neq 0\} \subset \bigcup_{n=1}^{\infty} \underbrace{\bigcup_{\Delta_k^n \subset U} A_k^n}_{B_n}$$

Note that $B_n \subset B_{n+1}$ since $A_k^n \subset A_{2k-1}^{n+1} \cup A_{2k}^{n+1}$

Claim: $m(B_n) < \varepsilon$ since $\Delta_k^n \subset U$ are all disjoint and

$$\left| \sum_{\Delta_k^n \subset U} c_k \chi_k^n \right|_0 < \varepsilon$$

for all choices of scalers.

Hence $m(\cup B_n) \leq \varepsilon$, i.e. $m(\Phi_i^{-1}(U) \cap \varphi_i(t) \neq 0\} \leq \varepsilon$

Now define

$$\tilde{T}f = \sum_{n=1}^{\infty} \varphi_i(t)\, f \circ \Phi_i(t)$$

This is a continuous linear operator on L_0

Claim: $\tilde{T} = T$

Enough to check: for almost all t, and all sufficiently large n, $\tilde{T}(\chi_{\Delta_k^n})(t) = T(\chi_{\Delta_k^n})(t)$ (where n depends on t) for all $k \le 2^n$.

Let t be given. Suppose $\varphi_1(t), \dots, \varphi_p(t) \neq 0$

Choose $n(t)$ so large that the distinct elements $\Phi_i(t)$ in this sequence all lie in disjoint intervals Δ_k^n, $n \ge n(t)$

$$\tilde{T}\chi_{\Delta_k^n}(t) = \underset{\substack{\uparrow \\ \text{by construction}}}{w_k^n(t)} = T\chi_{\Delta_n^k}(t)$$

$\boxed{/\!/}$

(Kalton)

<u>THEOREM:</u> Let $T: L_0 \to L_0$ be a non-zero continuous operator

Then T preserves a copy of L_0. There exists a measurable set A, $m(A) > 0$, such that $T: L_0(A) \to L_0$ is an isomorphism

Proof. Suppose $\exists\ \Phi: [0,1] \to [0,1]$ s.t. $m(A)=0 \Rightarrow m\,\Phi^{-1}(A) = 0$

When is $Tf = f \circ \Phi$ 1-1? Suppose $Tf = 0$ a.e. In particular suppose $T\chi_B = 0$ a.e. Then $\chi_{\Phi^{-1}(B)} = 0$ a.e., i.e. $m(\Phi^{-1}(B)) = 0$. This should imply $m(B) = 0$ if T is 1-1

We want a measurable subset A of $[0,1]$ s.t.

i) $m(A) > 0$

ii) if $m(\Phi^{-1}(B)) = 0$, then $m(B) = 0$ for any measurable subset B of A.

Let $f = \dfrac{d(m \circ \Phi^{-1})}{dm}$. Let $A = \{f > 0\}$. If $f = 0$ a.e. then $1 = m\,\Phi^{-1}[0,1] = \int_0^1 f = 0$ $\lightning$. Hence $m(A) > 0$

Also, $\int_B f\,dm = m(\Phi^{-1}(B)) = 0 \Rightarrow m(B) = 0$ for $B \subset A$.

Hence $\exists A$, $m(A) > 0$ such that on $L_0(B)$, T is 1-1

Want to check the B-continuity of T.

Suppose $Tf_n \to 0$. Does $f_n \to 0$? Equivalently,

if $T\chi_{B_n} \to 0$, does $\chi_{B_n} \to 0$, i.e. $m\,\Phi^{-1}(B_n) \to 0$

$\Rightarrow m(B_n) \to 0$? Yes, since $m \ll m \circ \Phi^{-1}$

9/24 L_p

(Proof of Kalton's thm cont.)

Have done case $Tf = \varphi f \circ \Phi$. In general, given $T \neq 0$, write

$$Tf = \sum \varphi_i \, f \circ \Phi_i$$

Let $\mathcal{F}$ be the class of all finite subsets of $\mathbb{N}$, $\forall F \in \mathcal{F}$ let

$$A_F := \{x : \varphi_i(x) \neq 0,\ i \in F,\ \varphi_j(x) = 0 \ \ j \notin F\}$$

The A_F's are measurable, they're pairwise disjoint and if $i, j \in F$

$$\Phi_i(x) < \Phi_j(x) \ \text{ if } i < j,\ x \in A_F$$

If $x \in A_F$,

$$Tf(x) = \sum_{i \in F} \varphi_i(x) \, f(\Phi_i(x))$$

$$\therefore \quad Tf = \sum_{F \in \mathcal{F}} \chi_{A_F} \sum_{i \in F} \varphi_i \, f \circ \Phi_i$$

Since $T \neq 0$, then for some F^*, $m(A_{F^*}) > 0$. There exists measurable subset C of A_{F^*}, $m(C) > 0$, and $n \in \mathbb{N}$ s.t.

i) $\Phi_i(C) \subset \Delta^n_{k_i}$ for some k

ii) $i \neq j \Rightarrow k_i \neq k_j$

Now look at one $\Delta^n_{k_i}$. $T|_{L_0(\Delta^n_{k_i})}$ If $f \in L_0(\Delta^n_{k_i})$ and $x \in A_{F^*}$, then

$$Tf(x) = \varphi_i(x) f \, \Phi_i(x)$$

Fact: If $\chi_C T|_A$ is an isomorphism, then $T|_A$ is an isomorphism

. Define ν on the subsets of $\Delta^n_{k_i}$ by

$$\nu(A) = m(\Phi^{-1}(A) \cap C)$$

$\nu \ll m$. If $h = \frac{d\nu}{dm}$, then h is not zero a.e. on Δ^n_k, so if $B = \{x \in \Delta^n_{k_i} : h(x) > 0\}$, then $m(B) > 0$ and

$$\chi_{\Phi^{-1}(B) \cap C} \, T|_{L_0(B)}$$

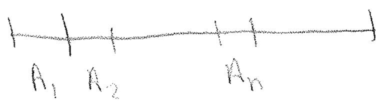
is an isomorphism

$\square$

Aside: Pelczynski Decomposition

Fact $\quad L_0 \simeq \prod_{i=1}^{\infty} L_0$ (product topology)

$\uparrow$ linear isomorphic

Proof. 

disjoint sets $\mu(A_i) > 0$

Then $L_0 = \prod_{n=1}^{\infty} L_0(A_n)$ (product topology is convergence in measure)

But $L_0(A_n) \simeq L_0(A)$.

We will prove that if $\mathfrak{X}$ is a complemented subspace of L_0, then there exists a complemented subspace Z of $\mathfrak{X}$ s.t. $Z \simeq L_0$

9/26 L_p

Any operator on L_0 to L_0 has form $Tf = \sum \varphi_i \, f \circ \Phi_i$

Suppose T also has representation $Tf = \sum \lambda_i \, f \circ \Lambda_i$. The uniqueness of representation of T: almost everywhere

$$\sum \varphi_i(x)\, \delta_{\Phi_i(x)} = \sum \lambda_i \delta_{\Lambda_i(x)}$$

Reason - Suppose $Tf = \varphi \, f \circ \Phi = \lambda \, f \circ \Lambda$. Then $\{\varphi \neq 0\} = \{\lambda \neq 0\}$

If $f \circ \Phi = f \circ \Lambda \ \forall f$, then $\chi_A \Phi = \chi_A \Lambda \Rightarrow \chi_{\Phi^{-1}(A)} = \chi_{\Lambda^{-1}(A)}$ for all A, so Φ and Λ agree almost everywhere.

FACT: A complemented subspace of ℓ_1 is $\simeq$ to ℓ_1

(1) $\ell_1 = (\sum \ell_1)_{\ell_1}$

(2) A complemented subspace of ℓ_1 contains a complemented subspace $\simeq \ell_1$

(3) Apply Pelzynski's decomposition

Pelczynski decomposition for L_0

$$L_0 \simeq \prod_{n=1}^{\infty} L_0$$

Suppose $\mathcal{X}$ is complemented in L_0, $\mathcal{Y}$ complemented in $\mathcal{X}$ and $\mathcal{Y} \simeq L_0$. Then $\mathcal{X} \simeq L_0$ for

$$L_0 \simeq \mathcal{X} \oplus \mathcal{Z} \simeq \prod_{n=1}^{\infty} (\mathcal{X} \oplus \mathcal{Z}) \qquad (\mathcal{X} = \mathcal{Y} \oplus W)$$

$$\simeq \prod_{n=1}^{\infty} (\mathcal{Y} \oplus W \oplus \mathcal{Z})$$

$$\simeq \prod_{n=1}^{\infty} (L_0 \oplus W \oplus \mathcal{Z})$$

$$\simeq \prod_{n=1}^{\infty} L_0 \oplus \prod_{n=1}^{\infty} W \oplus \prod_{n=1}^{\infty} \mathcal{Z}$$

$$\simeq \prod_{n=1}^{\infty} L_0 \oplus \prod_{n=1}^{\infty} W \oplus \prod_{n=1}^{\infty} \mathcal{Z} \oplus W$$

$$\simeq L_0 \oplus W$$

$$\simeq \mathcal{Y} \oplus W \simeq \mathcal{X}$$

LEMMA: Suppose $\mathcal{X}$ is an $\mathcal{F}$-space and $\mathcal{Y}$ (closed) subspace of $\mathcal{X}$

Suppose V is a continuous linear operator on $\mathcal{X}$ such that

i) $V|_{\mathcal{Y}}$ is a linear homeomorphism

(ii) $V(\mathcal{Y})$ is complemented in $\mathcal{X}$

Then $\mathcal{Y}$ is complemented in $\mathcal{X}$

Proof: Let $P: \mathcal{X} \to V(\mathcal{Y})$ be a projection. Let $U = (V|_{\mathcal{Y}})^{-1}$

The projection we need is $U \circ P \circ V$

$\blacksquare$

Suppose $\mathcal{X}$ is a complemented subspace of L_0

We will find a closed subspace $\mathcal{Y}$ of $\mathcal{X}$ and an operator V on L_0 such that $V(\mathcal{Y}) = L_0(A)$, $m(A) > 0$, and $V|_{\mathcal{Y}}$ is a linear homeomorphism. Then $\mathcal{Y}$ is complemented, so by Pelczynski decomposition $\mathcal{X} \simeq L_0$. Note: any $L_0(A)$ is complemented ($f \to \chi_A f$ is a projection)

If P is a projection operator

$$Pf = \sum \varphi_i \, f \circ \Phi_i$$

Then $P^2 f = \sum_{i=1}^{\infty} \varphi_i \sum_{j=1}^{\infty} (\varphi_j \circ \Phi_i)(f \circ \Phi_j \circ \Phi_i)$

$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varphi_i (\varphi_j \circ \Phi_i)\, f \circ (\Phi_j \circ \Phi_i)$

$=$ Kwapian representation of P^2

But $P^2 = P$, so

$$\sum_i \varphi_i(x)\, \delta_{\Phi_i(x)} = \sum_{i,j} \varphi_i(x)\, \varphi_j(\Phi_i(x))\, \delta_{\Phi_j(\Phi_i(x))} \quad \text{a.e.}$$

by the uniqueness of representation. Can find Φ_{i_1} s.t.

$$\Phi_{i_1}(x) = \Phi_{k_1} \Phi_{l_1}(x) = \Phi_{k_1} \Phi_{k_2} \Phi_m(x) = \dots = \Phi_{k_1} \Phi_{k_2} \cdots \Phi_{k_{l-1}} \Phi_{i_1}(x)$$

Then $\Phi_{k_{l-1}}$ is 1-1 on range Φ_{i_1}

9/29 L_p

$$\sum_i \varphi_i(x)\,\delta_{\Phi_i(x)} = \sum_{j,k} \varphi_j(x)\,\varphi_k\,\Phi_j(x)\,\delta_{\Phi_k(\Phi_j(x))}$$

We want a closed subset C of positive measure such that for some i, Φ_i is continuous on C, Φ_i is 1-1 on C, φ_i is non-zero on C

1. For each x $\exists$ $N(x)$ s.t. $\varphi_i(x) = 0$ $\quad i > N(x)$ and $\varphi_j(x)\,\varphi_k\,\Phi_j(x) = 0$ for $j,k > N(x)$. So there exists a set C_1 of positive measure such that $N(x) \le M$ $\forall x \in C_1$

2. If $\psi : \{1,2,\dots,M\} \longrightarrow \{1,2,\dots,M\}$, then for some i and k, $\psi^k(i) = i$

$i \in \{1,2,\dots,M\}$

3. For $x \in C_1$ such that $\varphi_i(x) \neq 0$ we have a $j(i)$ and $k(i)$ s.t. $\varphi_{j(i)}(x)\,\varphi_{k(i)}\,\Phi_{j(i)}(x) \neq 0$ and $\Phi_i(x) = \Phi_{k(i)} \circ \Phi_{j(i)}(x)$

$(\, j(i) \in \{1,\dots,M\}\,)$ From (2), $j^l(i) = i$ for som l and for some $i \in \{1, 2, \ldots, M\}$

$$\Phi_i(x) = \Phi_{k(i)} \Phi_{j(i)}(x)$$

$$= \Phi_{k_1(i)} \Phi_{k_2(i)} \Phi_{j(j(i))}(x)$$

$$\cdots$$

$$= \Phi_{k_1(i)} \cdots \Phi_{k_\ell(i)} \Phi_{j^\ell(i)}(x)$$

$$= \Phi_{k_1(i)} \cdots \Phi_{k_\ell(i)} \Phi_i(x)$$

4. On $C_2 \subset C_1$, $m(C_2) > 0$, $\Phi_i = \Phi_{k_1(i)} \circ \cdots \circ \Phi_{k_\ell(i)} \Phi_i$

$$\left.\begin{array}{l} \varphi_{k_\ell(i)} \circ \Phi_i \neq 0 \\ \varphi_i \neq 0 \end{array}\right\} \text{on } C_2$$

5. $\Phi_{k_\ell(i)}$ is 1-1 on $\Phi_i(C_2)$

6. There is a closed subset C_3 of C_2 such that Φ_i is continuous on C_3 $m(C_3) > 0$

7. $m(\Phi_i(C_3)) > 0$

8. $\exists$ closed subset C_4 of $\Phi_i(C_3)$, $m(C_4)>0$, such that $\Phi_{k_\ell}|_{C_4}$ is continuous and $\varphi_{k_\ell} \circ \Phi_i \neq 0$ on C_3 so $\varphi_{k_\ell} \neq 0$ on $\Phi_i(C_3)$, i.e. $\varphi_{k_\ell} \neq$ on C_4.

9. $\exists$ closed subset $C_5 \subset C_4$, $m(C_5)>0$ and an n such that on C_5 if some $\varphi_j(x)$ is non-zero, then either

$\Phi_j(x) = \Phi_{k_\ell(i)}(x)$ or $\Phi_j(C_5) \subset \Delta^n_q$

$\Phi_{k_\ell(i)}(C_5) \subset \Delta^n_r \quad r \neq q$

10. Sum up the φ_j on C_5 for which $\Phi_j = \Phi_{k_\ell}$. Call this sum φ_{k_ℓ}. May assume $\varphi_{k_\ell} \neq 0$

11. Consider $P|_{L_0(\Phi_{k_\ell}(C_5))}$. Define V on L_0 by

$$V(f)x = \begin{cases} \dfrac{1}{\varphi_{k_\ell}(x)} f(x) & x \in C_5 \\ 0 & x \notin C_5 \end{cases}$$

10/1 L_p

Define ν on measurable subsets of $\Phi_{L_0}(C_5)$ by

$$\nu(A) = m\left(\Phi_{L_0}^{-1}(A) \cap C_5\right)$$

Then $\nu \ll m$. Let $h = d\nu/dm$. Let $C_6 = \{x \in \Phi_{L_0}(C_5) : h > 0\}$

$p|_{C_0(C_6)}$ is a linear homeomorphism

$$\text{If } \quad Vf(x) = \begin{cases} \dfrac{1}{\varphi_{L_0}(x)} f(x) & x \in \Phi_{L_0}^{-1}(C_6) \cap C_5 \\ 0 & \text{otherwise} \end{cases}$$

then $V \circ p|_{L_0(C_6)} = L_0\left(\Phi_{L_0}^{-1}(C_6) \cap C_5\right)$. So $V \circ P L_0(C_6)$ is complemented

Further, V is a linear homeomorphism on $PL_0(C_6)$. By the earlier

lemma, $PL_0(C_0)$ is complemented.

$\square$

Suppose $T: L_0(\mathcal{A}, m) \to L_0(\mathcal{B}, m)$ operator where

$\mathcal{B}$ is a sub-algebra of $\mathcal{A}$. By Kwapian

$$Tf = \sum_i \varphi_i \, f \circ \Phi_i$$

Claim: φ_i is $\mathcal{B}$-$\mathbb{R}$ measurable
Φ_i is $\mathcal{B}$-$\mathbb{R}$ measurable

$$\Phi_i^n = \inf\{k : \sum_{j=1}^{k} \chi_j^n \geq i\} / 2^n$$

$$\chi_j^n = \operatorname{supp} T \underbrace{\chi_{\Delta_j^n}}_{\mathcal{B} \text{ measurable}}$$

$\therefore \Phi_i^n$ is $\mathcal{B}$-$\mathbb{R}$ measurable

Now $\varphi_i^n = w_k^n$ if $\Phi_i^n = k/2^n$ so φ_i^n is $\mathcal{B}$-$\mathbb{R}$ measurable.

Hence Φ_i, φ_i are $\mathcal{B}$-measurable

Note: Kwapian theorem holds for all separable non-atomic

measure spaces

COROLLARY: There is no projection from $L_0(I^2, \mathcal{A})$ onto $L_0(I^2, \mathcal{V})$, where $\mathcal{V}$ is the subalgebra generated by vertical strips

Proof. Suppose there were such a projection P

$$Pf(z) = \sum_i \varphi_i(z)\, f\circ \Phi_i(z)$$

where $\Phi_i^{-1}(B)$ for each Borel set B is a vertical strip $(\Phi_i : I^2 \to I^2)$

From the preceding proof (C_5 or C_6) there exists a measurable C, $m^2(C) > 0$ s.t. for some particular i

a) Φ_i is 1-1 on C

b) $m^2\, \Phi_i(C) > 0$

c) $\forall$ Borel subset B of C, $B = \Phi_i^{-1}(D) \cap C$

where D is a Borel subset of $\Phi_i(C)$

i.e. for any Borel set B, $B \cap C = \Phi_i^{-1}(D) \cap C$ for some D.

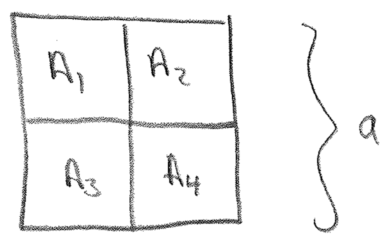
Hence $\exists\, C$ Borel, $m^2(C) > 0$ s.t. $\forall$ Borel B,

$B \cap C = (\text{vertical strip}) \cap C$, C can't be a square

Claim: no such C exists (Peck)

By Vitali's theorem, $\exists$ small square S s.t. $m^2(S \cap C) > \frac{1}{2} m^2(S)$

S:



Let $B_1 = A_1 \cup A_4$ $\quad B_2 = A_2 \cup A_3$. Then

$$\text{disjoint} \begin{cases} B_1 \cap C = V_1 \cap C \\ B_2 \cap C = V_2 \cap C \end{cases}$$

where $V_1 = R_1 \times I \qquad R_1 = P_x(B_1 \cap C)$

$V_2 = R_2 \times I \qquad R_2 = P_x(B_2 \cap C)$

Note $R_1 \cap R_2 = \emptyset$. A point in common is projection of a point in $(B_1 \cap C) \cap (B_2 \cap C)$ $\lightning$

$$m^2(S \cap C) = \int_{R_1} m(S \cap C)_x \, dm(x) + \int_{R_2} m(S \cap C)_x \, dm(x)$$

$$= \int_{R_1' \cup R_1''} m(S \cap C)_x \, dm(x) + \int_{R_2' \cup R_2''} m(S \cap C)_x \, dm(x)$$

$P_x(A_1 \cap C)$ $\quad$ $P_x(A_4 \cap C)$

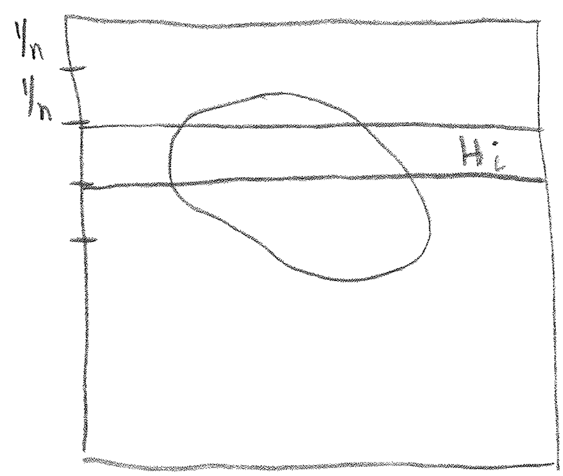
$$\leq a \cdot \frac{1}{2} a = \frac{a^2}{2}$$

since by construction $> \frac{1}{2} a^2$

10/3 L_p

Fact (J. Kupka) $\exists$ no measurable C in I^d, $m^d(c) > 0$ s.t. $\forall$ Borel sets B $\exists$ a vertical strip V s.t. $B \cap C = V \cap C$ up to sets of measure 0

Proof. Let $n \in \mathbb{N}$. Partition I into intervals of length $1/n$



$H_i = I \times I_i$

For each i $\exists$ vertical strip V_i s.t. $H_i \cap C = V_i \cap C$. We may assume that $V_i = \underbrace{P_x(H_i \cap C)}_{R_i} \times I$ (measurable)

Claim: $(R_i \cap R_j) = \emptyset$

$$m^d(C) = \sum_i m^2(C \cap H_i) = \sum_i \int m((C \cap H_i)_x)\, dm(x)$$

$$= \sum_i \int m\big(((C \cap V_i))_x\big)\, dm(x)$$

$$= \sum_i \int_{R_i} m(C \cap V_i)_x \, dm(x) \le \sum_i \tfrac{1}{n} m(R_i) \le \tfrac{1}{n}$$

$\blacksquare$

Further remarks on L_0

(1) OPEN PROBLEM (Pelczynski) Suppose $E \subset L_0$ subspace s.t. in the relative topology E is locally convex. Then is there a lifting theorem for L_0/E?, i.e.

$$\begin{tikzcd} L_0 \arrow[r, "T"] \arrow[dr, dashed, "\tilde{T}"'] & L/E \\ & L \arrow[u, "\pi"'] \end{tikzcd}$$

If $E =$ space ω of all sequences, then there is a lifting

countable product of real lines

(2) If $T: L_0 \longrightarrow \mathfrak{X}$ (F-space) is non-zero, must T preserve a copy of L_0? No (Kalton)

(3) If $T: L_0 \longrightarrow \mathfrak{X}$ is non-zero, T does preserve a copy of ℓ_2

(4) Open problem - characterize the Banach spaces which embed in L_0. Conjecture (Rosenthal) Any such B-space embeds in L_1

It is known that any such B-space embeds in

$$\text{weak } L_1 = \{ f : \sup_{c>0} c\, m\{|f|>c\} < \infty \}$$

(This space is p-convex for $p<1$ (Kalton) but not locally convex)

(5) What is $L_0(I^2) \Big/ \text{Functions constant on vertical strips}$?

Is this isomorphic to L_0?

L_p $\quad p > 0$

Proposition: $\exists$ no positive projection of $L_p(I^2)$ onto functions contant on vertical lines

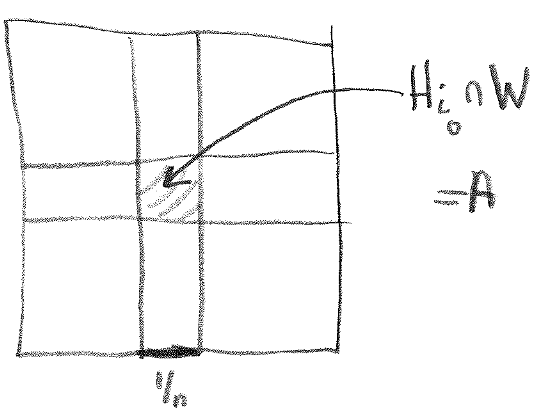
Proof. Suppose $\exists$ such a projection P. Given $n > N$. Take the usual horizontal strips H_i, $1 \le i \le n$. Then

$$\sum_i P\chi_{H_i} = P1 = 1$$

so for at least one i

$$m^2\{|P\chi_{H_{i_0}}| \ge 1/n\} \ge 1/n$$

Let $V = \{z : |P\chi_{H_{i_0}}(z)| \ge 1/n\}$. Let W be a vertical substrip with $m^2(W) = 1/n$.



$$m^2(A) = 1/n^2$$

$P\chi_A \ge 1/n$ on a set of measure $1/n$

$$\frac{\| P \chi_{A} \|_p}{\| \chi_{A} \|_p} \geq \frac{\left(\frac{1}{n^p} \frac{1}{n} \right)^{1/p}}{\left(\frac{1}{n^2} \right)^{1/p}} = n^{(1-p)/p} \xrightarrow[n \to \infty]{} \infty$$

$\therefore$ P is not continuous

10/6 L_p

Revision of Kupka's proof (removing use of projections)

Assume $H_i \cap C = V_i \cap C$. Consider $\{V : H_i \cap C = V \cap C\} = \mathcal{V}_i$

Let (V_j) be dense in $\mathcal{V}_i$. Let $V_i^* = \bigcap_j V_j$ (measurable). Then

$H_i \cap C = V_i^* \cap C$. Also, if $H \cap C = V \cap C$, then $m^*(V_i^* \setminus V) = 0$

If $H_i \cap H_j = \phi$, then the corresponding V_i^*'s are disjoint

Jon's ↑

Mine ↓

FACT: Suppose $\ell_p^n \xrightarrow{T} \mathfrak{X}/N$ where $\mathfrak{X}$ is p-convex space

↑ closed subspace

$\{(a_i)_{i=1}^n\}$ with quasi-norm $\left(\sum |a_i|^p\right)^{1/p}$

Then T can be lifted

$$\begin{tikzcd} \ell_p^n \arrow[r, "T"] \arrow[dr, dashed, "\tilde{T}"'] & \mathfrak{X}/N \\ & \mathfrak{X} \arrow[u, "\pi"'] \end{tikzcd}$$

Moreover, $\|\tilde{T}\| \leq 2 \|T\|$

Proof. Look at the basis vectors (e_i) of ℓ_p^n. For each i pick $x_i \in \mathfrak{X}$ s.t. $\|x_i\| \leq 2\|\pi(x_i)\|$ and $\pi(x_i) = Te_i$

F norm in $\mathfrak{X}$

Define $\tilde{T}(e_i) = x_i$ and extend by linearity. Then

$$\|\tilde{T}\sum \alpha_i e_i\|^p = \|\sum \alpha_i \tilde{T}e_i\|^p \leq \sum |\alpha_i|^p \|\tilde{T}e_i\|^p$$

$$\leq \sum |\alpha_i|^p \|x_i\|^p$$

$$\leq \sum |\alpha_i|^p 2^p \|T\|^p$$

So $\|\tilde{T}\sum \alpha_i e_i\| \leq \left(\sum |\alpha_i|^p\right)^{1/p} 2\|T\|$

$\square$

<u>DEFINITION:</u> A p-convex space $\mathfrak{X}$ is a $\mathcal{L}_p$ space if

$$\mathfrak{X} = \overline{\bigcup_\alpha \mathfrak{X}_\alpha}$$

Where (i) $\alpha \leq \beta \Rightarrow \mathfrak{X}_\alpha \subset \mathfrak{X}_\beta$

(ii) If $k_\alpha = \dim \mathfrak{X}_\alpha < \infty$, then $\forall \alpha$, $\exists$ $\underset{\text{isomorphism}}{S_\alpha} : \mathfrak{X}_\alpha \xrightarrow{\text{onto}} \ell_p^{k_\alpha}$

s.t. $\|S_\alpha\| \|S_\alpha^{-1}\| \leq C$ (C independent of α)

Examples

(1) ℓ_p $\quad \mathfrak{X}_n = \{(x_i) : x_i = 0 \ \ i \geq n+1\}$

(2) L_p Given n partition $[0,1]$ into 2^n pieces , $A = \mathbb{N}$

$$\Big\| \sum \alpha_i \chi_{\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right]} \Big\|_p = \frac{1}{2^{n/p}} \Big(\sum_i |\alpha_i|^p\Big)^{1/p}$$

looks like ℓ_p norm

(put $2^{n/p}$ into def. of S_α)?

Span of simple functions over dyadic intervals dense in L_p

(3) H_p is not $\mathcal{L}_p$

Lifting theorem: Let $\mathcal{Y}$ be a $\mathcal{L}_p$-space. Let $\mathfrak{X}$ be p-convex and suppose N is a closed subspace of $\mathfrak{X}$ whose unit ball is compact in some Hausdorff vector topology. Then if $T : \mathcal{Y} \to \mathfrak{X}/N$, there is a lifting $\tilde{T} : \mathcal{Y} \to \mathfrak{X}$.

Remark: This applies when N is finite dimensional or when $\mathfrak{X}$ is L_p of circle and N is H_p (unif. conv. on compact sets of $|z|<1$ - compact top for unit ball of H_p)

$$\mathcal{Y} = \overline{\bigcup \mathcal{Y}_\alpha}$$

Proof (a) For each α there is a lift of $\tilde{T}_\alpha : \mathcal{Y}_\alpha \longrightarrow \mathcal{X}$

with $\|\tilde{T}_\alpha\| \le 2c\|T\|$

For $y \in \mathcal{Y}$, let $u(y) = z$, where $\pi(z) = Ty$ and

$\|u(y)\| \le 2\|Ty\| \le 2\|T\|\|y\|$. Consider

$$\prod_{y \in \bigcup_\alpha Y_\alpha} K_y B_N$$

K_y: scalar; B_N: unit ball of N

This is a compact set. For $\alpha \in A$, define f_α in this space

$$f_\alpha(y) = \begin{cases} \tilde{T}_\alpha(y) - u(y) & \text{if } y \in \mathcal{Y}_\alpha \text{ (ball there of)} \\ 0 & \text{if } y \notin \mathcal{Y}_\alpha \end{cases}$$

$\pi(\tilde{T}_\alpha(y) - u(y)) = 0$ so $\tilde{T}_\alpha(y) - u(y) \in N$

$$\|\tilde{T}_\alpha(y) - u(y)\| \le 2^{1/p-1}(2c\|T\|\|y\| + 2\|T\|\|y\|)$$

9/8 L_p

(Change $\prod_{\cup B(Y_\alpha)}$ to $\prod_{\cup Y_\alpha}$)

$$K_y > 2^{1/p-1}(2C+2)\|T\|\,\|y\|$$

Let f_β be a convergent subnet of f_α.

$$f_\beta(y) = \tilde{T}_\alpha(y) - u(y)$$

for $\beta \geq \alpha$, where $y \in Y_\alpha$. Let $f_\beta(y) \to v(y)$

$$f_\beta(y) + u(y) \longrightarrow v(y) + u(y)$$

$$\therefore \tilde{T}_\beta(y) \to v(y) + u(y)$$

i.e. $\tilde{T}_\beta$ is pointwise convergent to $\tilde{T}$ defined on $\bigcup_{\alpha\in A} Y_\alpha$

$\tilde{T}$ is linear since range is Hausdorff.

$$\|\tilde{T}\| \leq 2C\|T\|$$

This makes $\tilde{T}$ continuous on $\bigcup_{\alpha\in A} Y_\alpha$ and so $\tilde{T}$ extends to a continuous linear $\tilde{T}$ defined on Y with $\|\tilde{T}\| \leq 2C\|T\|$. ∎

(complete)

THEOREM: Let X be p-convex, let N be a closed subspace of X and let $T: L_p \longrightarrow X/N$. If equasi-norm on N is equivalent to a q-convex quasi-norm for $q > p$, then there is a unique lift $\tilde{T}$

Proof. Uniqueness - We show if $U: L_p \longrightarrow Z$ (q-convex) $(q>p)$ is continuous, then $U = 0$

Suppose $f \in L_p$, $|f| \leq M$ a.e. For any n

$$f = \sum_{k=1}^{n} f \cdot \chi_{[\frac{k-1}{n}, \frac{k}{n}]}$$

$$\Rightarrow Uf = \sum_{k=1}^{n} Uf \cdot \chi_{[\frac{k-1}{n}, \frac{k}{n}]}$$

$$\therefore \|Uf\|^q \leq \sum_{k=1}^{n} \|U\|^q \|f \chi_{[\frac{k-1}{n}, \frac{k}{n}]}\|_p^q$$

$$\leq \|U\|^q M^q n^{-p/q} n = \|U\|^q M^q n^{1-p/q} \longrightarrow 0$$

Thus $U = 0$

Proof of existence

Notation: $\chi^n_j = \chi_{\Delta^n_j}$ $\quad \Delta^n_j = [\frac{j-1}{2^n}, \frac{j}{2^n}]$

$L_p = \overline{\bigcup_n Y_n}$ $\quad Y = sp\, \chi^n_j$ $\quad j = 1, 2, \ldots, 2^n$

For each n, there is a lift $\tilde{T}_n$ of $T|_{Y_n}$ with

$\|\tilde{T}_n\| \leq 2\|T\|$.

Claim: The sequence $(\tilde{T}_n(\chi^m_j))_{n=m}^{\infty}$ is Cauchy in n

for fixed j and fixed m. Take $n_1 > n_2 \geq m$

$\|T_{n_1}(\chi^m_j) - T_{n_2}(\chi^m_j)\|$

(Write $\chi^m_j = \sum_{L=1}^{2^{n_2-m}} \chi^{n_2}_{q_{i,j}}$)

$$= \| \sum_{L=1}^{2^{n_2-m}} \underbrace{(T_{n_1} - T_{n_2})\chi^{n_2}_{q_{i,j}}}_{\text{These all lie in } N \text{ which is } q\text{-convex}} \|$$

$$\le K\left(\sum_{L=1}^{2^{n_2-m}} \|(T_{n_1}-T_{n_2})\chi^{n_2}_{q_{i,j}}\|^q\right)^{1/q}$$

$$\le K\cdot\|T_{n_1}-T_{n_2}\|\left(\sum_{L=1}^{2^{n_2-m}} \|\chi^{n_2}_{q_{i,j}}\|_p^q\right)^{1/q}$$

$$= K\cdot\|T_{n_1}-T_{n_2}\|\left((2^{n_2})^{-q/p}\cdot 2^{n_2-m}\right)^{1/q}$$

$$= K\|T_{n_1}-T_{n_2}\|\left(2^{n_2(1-q/p)}\,2^{-m}\right)^{1/q}$$

$$\le K\cdot 4\cdot 2^{1/p-1}\,\|T\|\left(2^{n_2(1-q/p)}\,2^{-m}\right)^{1/q}$$

$$\longrightarrow 0 \text{ as } n_2\to\infty$$

10/10 L_p

$$f_\alpha(y) = \begin{cases} \tilde{T}_\alpha(y) - u(y) & y \in Y_\alpha \\ 0 & y \notin Y_\alpha \end{cases}$$

$V =$ limit of f_β

$$f_\beta(x+y) - f_\beta(x) - f_\beta(y) \qquad x, y \in Y_\beta$$

$$= \tilde{T}_\beta(x+y) - u(x+y) - \tilde{T}_\beta(x) + u(x) - \tilde{T}_\beta(y) + u(y)$$

$$= -u(x+y) + u(x) + u(y)$$

$\therefore \; V(x+y) - V(x) - V(y) = u(x) + u(y) - u(x+y)$

$\therefore \; \tilde{T}(y) := u(y) + V(y)$ is additive

and $\|\tilde{T}(y)\| \leq 2^{1/p-1}\left(2\|T\|\|y\| + 2^{\frac{1}{p}-1}\left(2C\|T\|\|y\| + 2\|T\|\|y\|\right)\right)$

Uniqueness of $\tilde{T}$: If $\tilde{T}$ and T are lifts, then $U: \tilde{T} - T_1 : L_p \to N$

Consider $U : L_p \to (N, \tau)$

$\uparrow$ compact top.

Claim: U is norm-τ cont. WLOG $\|U\| = 1$. Now

$U(B(L_p)) \subset B(N)$. Let W be a τ-nbhd of 0. Choose

$\lambda > 0$ so that $\lambda B(N) \subset W$. So $U(\lambda B(L_p)) \subset \lambda B(N) \subset W$

This shows claim

But then $U = 0$ since $U : L_p \to (N, \tau)$ is continuous and compact. If $U \neq 0$, then it preserves a copy of ℓ_2 (Kalton) which is impossible for a compact operator.

$\boxtimes$

Applications of Liftings

① $\exists T : H_p \xrightarrow{\text{onto}} \ell_p \qquad 0 < p \leq 1$

$$Tf = \left((1 - |z_n|^2)^{1/p} f(z_n) \right)$$

where (z_n) in $|z| < 1$ is a "uniformly separated" seq (See Duren)

Let $N = \ker T$. With the topology of unif. convergence on compact sets, the ball of N is compact since the ball of H_p is compact

$\therefore$ isomorphism

$$\ell_p \xrightarrow{S} H_p / N$$

has a lift $\tilde{S}$ from ℓ_p to H_p. Then ℓ_p is a complement subspace of H_p.

(2) Suppose X is a Hausdorff TVS and R is a 1-dimensional subspace of X such that $X/R \simeq \ell_p$

$$\ell_p \xrightarrow{T} X/R$$

~~Need X p-convex locally bdd to apply lifting theorem~~

If X was locally bounded and was p-convex, we could apply either lifting theorem (will show later this is the case)

(If X/B and B are both locally bdd, then X is locally bounded)

(If X/B is p-convex and B is q-convex, $q > p$, then X is p-convex)

L_0, ℓ_p $0<p<1$ are K-spaces, but ℓ_1 is not a K-space

i.e. $\exists\ X/_{\mathbb{R}} \simeq \ell_1$ yet $X \not\simeq \ell_1$ (Ribe)

ℓ_p, $p>1$ is a K-space.

L_p, $0<p<1$ are K-spaces $\quad$ L_1 not a K-spaces
$\quad\quad p>1$

Is c_0 a K-space? (Open)

$$\left(X/_{N} \simeq Y \Rightarrow X \simeq Y \right)$$

$\uparrow$ 1 dimensional subspace

Y K-space if above true $\forall X$

Operators on L_p (Kalton)

An operator on L_0 has the form $Tf(x) = \int f\, d\nu_x$ a.e.

$$= \sum \varphi_i(x) f\, \Phi_i(x)$$

$$= \int f\, d\left(\sum_i \varphi_i(x)\, \delta_{\Phi_i(x)}\right)$$

Operators on ℓ_p $(0<p\leq 1)$

$$T(e_i) = \sum_{j=1}^{\infty} \alpha_{ij} e_j$$

where $\left(\sum_j |\alpha_{ij}|^p\right)^{1/p} \leq \|T\|$

Claim: (a) $T\left(\sum_{i=1}^{\infty} c_i e_i\right) = \sum_j \sum_i c_i \alpha_{ij} e_j$

(b) $\|T\| = \sup_i \left(\sum_j |\alpha_{ij}|^p\right)^{1/p}$

Show sum in (a) is convergent

$$\sum_j \left|\sum_i c_i \alpha_{ij}\right|^p$$

10/13 L_p

Operators on ℓ_p

Given a sequence (a_{ij}) with $\sup_i \overset{(*)}{\left(\sum_j |a_{ij}|^p \right)^{1/p}} < \infty$

one can define an operator on ℓ_p by

$$T\left(\sum c_i e_i\right) = \sum_i \sum_j c_i a_{ij} e_i$$

Every operator on ℓ_p arises this way and $\|T\| = (*)$

$$j^{th} \text{ component} = \sum_i c_i a_{ij}$$

$$\left| \sum_i c_i a_{ij} \right| \leq \sum |c_i| \sup |a_{ij}|$$

$$\leq \left(\sum |c_i|^p \right)^{1/p} \sup |a_{ij}|$$

so j^{th} component makes sense

For continuity

$$\sum_j \left| \sum_i c_i a_{ij} \right|^p \leq \sum_j \sum_i |c_i|^p |a_{ij}|^p$$

$$= \sum_i \sum_j |c_i|^p |a_{ij}|^p = \sum_i |c_i|^p \sum_j |a_{ij}|^p$$

$$\leq \sum_i |c_i|^p \, (*)^p$$

$$\therefore \ \|T\|^p \leq (*)^p$$

The reverse is clear since $\|T\| \geq \|Te_i\| \quad \forall i \Rightarrow \|T\| > (*)$.

Operators on L_p $(0<p<1)$

Suppose λ is a finite measure on $(X, \mathcal{a})$. Define

$$|\lambda|_p(A) = \sup\left\{ \sum_{i=1}^{\infty} |\lambda(A_i)|^p : (A_i) \text{ partition of } A \right\}$$

Facts: (1) $|\lambda|_p$ is a countably additive measure on $\mathcal{a}$

(2) Suppose $|\lambda|_p(X) < \infty$. Then λ is purely atomic, with countably many atoms.

Write $\lambda = \lambda^+ - \lambda^-$. If λ is not purely atomic, WLOG λ^+ is not purely atomic so $\exists A_0 \in \mathcal{a}$ s.t. $\lambda^+|_{A_0}$ is atomless.
$\lambda^+(A_0) > 0$

Given n, by a standard exhaustion argument, can find sets

$A_1, \dots, A_n$ in A_0 such that $\lambda^+(A_i) = \dfrac{\lambda^+(A_0)}{n}$. Then

$$|\lambda|_p(X) \geq n\left(\lambda^+(A_0)\right)^p n^{-p} = n^{1-p}\, \lambda^+(A_0)^p \to \infty$$

$$\therefore\ |\lambda|_p(X) = \infty \ \lightning$$

We will be representing operators on L_p of the Cantor set.

Notation:
- Δ Cantor set
- $\mathcal{B}$ Borel sets in Δ
- $C(\Delta)$ cont. functions on Δ
- $\mathcal{M}(\Delta)$ regular Borel measures on Δ $= C(\Delta)^*$
- $\mathcal{W}^*$ weak*-Borel sets in $\mathcal{M}(\Delta)$
- μ normalized Haar measure on Δ

Suppose $\{\nu_x : x \in \Delta\}$ is a family of measures in $\mathcal{M}(\Delta)$ s.t.

(i) $x \to \nu_x$ is $(\mathcal{B}, \mathcal{W}^*)$ measurable

(ii) $M = \sup\limits_{\mu(B)>0} \dfrac{1}{\mu(B)} \displaystyle\int_X |\nu_x|_p(B)\, d\mu(x) < \infty$

Define an operator T on $\mathcal{B}$ simple functions by

$$T\left(\sum_{i=1}^n \alpha_i \chi_{B_i}\right)(x) = \int \sum \alpha_i \chi_{B_i}\, d\nu_x$$

$$= \sum_{i=1}^{n} \alpha_i \nu_x(B_i)$$

Claim: (Assuming measurability + well-defined) $\|T\| \leq M^{1/p}$

$$\int \left| \sum_{i=1}^{n} \alpha_i \nu_x(B_i) \right|^p d\mu(x) \leq \int \sum_{i=1}^{n} |\alpha_i|^p |\nu_x(B_i)|^p \, d\mu(x)$$

$$= \sum_{i=1}^{n} \frac{1}{\mu(B_i)} \int |\alpha_i|^p |\nu_x(B_i)|^p \mu(B_i) \, d\mu(x)$$

$$\leq \sum_{i=1}^{n} |\alpha_i|^p M \mu(B_i) = M \|f\|_p^p$$

Lemma: $\lambda \longmapsto |\lambda|$ is W^*-W^* measurable

Proof. Let $0 \leq f \in C(\Delta)$. $\{\tau : \tau(f) \leq K\} \in W^*$

Let (f_i) be dense in $[0, f]$. So

$$|\lambda|(f) \leq K \iff \lambda(f_i) \leq K \quad \forall i$$

$$\therefore |\lambda|(f) \leq K \iff \lambda \in \bigcap_i \{\tau(f_i) \leq K\} \in W^*$$

10/15 L_p

$$\Delta = \prod_{i=1}^{\infty} \{0,1\} \quad \text{with} \quad \lambda(\{0\}) = 1/2 \quad \lambda(\{1\}) = 1/2 \quad \text{and product measure } \mu$$

<u>LEMMA:</u> $\lambda \longrightarrow |\lambda|$ is W^*-W^* measurable

Proof (i) $f \geq 0$ Take (f_i) dense in $[0,f]$. Then for any measure λ, $|\lambda|(f) = \sup_i \lambda(f_i)$. Then

$$|\lambda|(f) \leq K \iff \lambda(f_i) \leq K \quad \forall i$$

$$\therefore \{\lambda : |\lambda|(f) \leq K\} = \bigcap_i \{\lambda : \lambda(f_i) \leq K\}$$

(ii) Write $f = f^+ - f^-$. $|\lambda|(f) = |\lambda|(f^+) - |\delta|(f^-)$

Then $\{\lambda : |\lambda|(f) \leq K\} = \{\lambda : |\lambda|(f^+) - |\lambda|(f^{-1}) \leq K\}$

$$= \{\lambda : \sup_i \lambda(f_i) - \sup_j \lambda(g_j) \leq K\} \quad \begin{array}{l}(f_i) \text{ dense in } [0,f^+] \\ (g_j) \text{ dense in } [0,f^-]\end{array}$$

If $(\alpha_i), (\beta_j)$ reals

$$\sup \alpha_i - \sup \beta_j \leq K \iff \sup \alpha_i \leq \sup \beta_j + K$$

$$\iff \sup \alpha_i \in \bigcap_n \bigcup_j \{x : x \leq \beta_j + 1/n + K\}$$

Now take $\alpha_i = \lambda(f_i)$, $\beta_j = \lambda(g_j)$, so

$$\{\lambda : |\lambda|(f) \leq K\} = \bigcap_n \bigcup_j \bigcap_i \left\{\lambda : \lambda(f_i) \leq \lambda(g_j) + \tfrac{1}{n} + K \right\}$$
$$\in W^*$$

Thus $\{\lambda : |\lambda|(f) \leq K\}$ is W^*-Borel, but these sets generate the W^*-Borel sets. $\boxed{\checkmark}$

LEMMA. If $x \longrightarrow \nu_x$ is $\mathcal{B}$-W^* measurable, then for any Borel set B, $x \longrightarrow \nu_x(B)$ is $\mathcal{B}$-measurable

Proof By the above, we know that $x \longrightarrow |\nu_x|$ is $\mathcal{B}$-W^* measurable

Let $\mathcal{C} = \{ B \in \mathcal{B} : x \longrightarrow |\nu_x|(B)$ is $\mathcal{B}$-measurable $\}$

(i) $\mathcal{C}$ contains all clopen sets (since χ_B is continuous if B is clopen

(ii) The clopen sets are a ring.

(iii) $\mathcal{C}$ is a monotone class ($B_n \uparrow B \Rightarrow B \in \mathcal{C}$
$B_n \downarrow B \Rightarrow B \in \mathcal{C}$)

By Halmos' theorem, (Monotone class th^{m}), $\mathcal{C}$ contains the σ-ring generated by the ring of clopens, i.e. $\mathcal{C}$ contains $\mathcal{B}$

In the same way $x \to (|\nu_x| - \nu_x)(B)$ is $\mathcal{B}$-measurable so that $x \to \nu_x(B)$ is $\mathcal{B}$-measurable $\forall B$

$\blacksquare$

<u>LEMMA</u>: (i) $\lambda \to |\lambda|_p(\Delta)$ is w^* lower semi-continuous

(ii) $\lambda \to |\lambda|_p(U)$ is w^* l.s.c. for any open U

Proof (i) $|\lambda|_p(\Delta) = \sup\limits_{\substack{C_i \text{ clopen} \\ C_i \cap C_j = \phi}} \sum\limits_{i=1}^{n} |\lambda(C_i)|^p$ $\leftarrow$ sup of w^* continuous functions

For suppose (A_i) is a Borel partition of Δ. Consider A. Given ε can find $K \subset A$ compact s.t. $|\lambda|(A \Delta K) < \varepsilon/3$ and can find open $U \supset K$ s.t. $|\lambda|(U \sim K) < \varepsilon/3$. $\exists$ clopen V, $K \subset V \subset U$

Then $|\lambda|(A \Delta V) < \varepsilon$

10/17 L_p

LEMMA: Assuming $x \longrightarrow \nu_x$ is $\mathcal{B}$-$\mathcal{W}^*$ measurable

(i) $x \longrightarrow |\nu_x|_p(\Delta)$ is $\mathcal{W}^*$-l.s.c.

(ii) $x \longrightarrow |\nu_x|_p(U)$ is $\mathcal{W}^*$-l.s.c. (U open)

(iii) If $|\nu_x|_p(\Delta) < \infty$ $\forall x$ in $\Delta_0 \in \mathcal{B}$, then $x \longrightarrow |\nu_x|_p(B)$ is

B-measurable if we look only at $x \in \Delta_0$

Proof (i) OK

(ii) Claim: $$|\nu_x|_p(U) = \sup_{\substack{C_i \text{ open in } U \\ C_i \cap C_j = \emptyset \ i \neq j}} \left\{ \sum_{i=1}^{n} |\nu_x(C_i)|^p \right\}$$

$= \sup$ continuous functions

(iii) Let $\mathcal{D} = \{ B \in \mathcal{B} : x \to |\nu_x|_p(B)$ is measurable, $x \in \Delta_0 \}$

$\mathcal{D}$ contains clopens from (ii) and as before $\mathcal{D}$ is monotone class

($D_n \downarrow D \Rightarrow |\nu_x|_p(D_n) \downarrow |\nu_x|_p(D)$ since $|\nu_x|_p(D_1) < \infty$, $x \in \Delta_0$)

$\boxtimes$

<u>LEMMA</u>: If $\nu_x^n \xrightarrow{w^*} \nu_x$ $\forall x$ and if $x \to \nu_x^n$ is $\mathcal{B}$-W^* measurable, then $x \to \nu_x$ is $\mathcal{B}$-W^* measurable

Define $T\big(\underbrace{\sum \alpha_i \chi_{A_i}}_{f}\big) = \sum \alpha_i \nu_x(A_i) = \int f \, d\nu_x$ as before

This is Borel measurable and $\|T\| \leq M^{1/p}$. T is defined on a dense linear subspace of $L_p(\Delta, \mathcal{B}, \mu) \longrightarrow L_p(\Delta, \mathcal{B}, \mu)$. So T extends to an operator $L_p(\Delta, \mathcal{B}, \mu) \longrightarrow L_p(\Delta, \mathcal{B}_\mu, \mu)$

Now given $T: L_p(\Delta, \mathcal{B}, \mu) \to L_p(\Delta, \mathcal{B}, \mu)$. We want to show that T comes from a map $x \to \nu_x$ as before

Computation: Suppose $\nu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$. Then

$$|\nu|_p(B) = \sum_{x_i \in B} |\alpha_i|^p$$

For $1 \le k \le 2^n$, write $k = \sum_{i=1}^{n} \varepsilon_i 2^i$ $\quad (\varepsilon_i = 0$ or $1)$

$$\Delta_k^n = \{x \in \Delta : x_i = \varepsilon_i \text{ (from } k), \ 1 \le i \le n\} \quad \text{clopen set}$$

$$\Delta_k^n = \Delta_{2k-1}^{n+1} \cup \Delta_{2k}^{n+1}$$

$$W_k^n = T\chi_{\Delta_k^n}$$

Then $x \longrightarrow W_k^n(x)$ is $\mathcal{B}$-measurable

Computation: Suppose $\delta \in \mathcal{M}(\Delta)$. Suppose $h(x)$ is Borel measurable on Δ

Then $x \longrightarrow h(x)\delta$ is $\mathcal{B}$-W^* measurable

For each n,k fix $\tau_k^n \in \Delta_k^n$ $\left(\cancel{\text{satisfying } \delta_{\tau_k^n} = \delta_{\tau_{2k-1}^{n+1}} + \delta_{\tau_{2k}^{n+1}}}\right)$ (?)

Define

$$b_k^n(x) = w_k^n(x)\,\delta_{\tau_k^n}$$

This is $\mathcal{B}$-W^* measurable. Set

$$\nu_x^n = \sum_{k=1}^{2^n} b_k^n(x)$$

$x \longrightarrow \nu_x^n$ is $\mathcal{B}$-W^* measurable

Note

$$W_k^n = W_{2k-1}^{n+1} + W_{2k}^{n+1} \quad \text{a.e.}$$

Off a set of measure 0, all such identities will hold everywhere.

$$T\chi_{\Delta_k^n}(x) = \int \chi_{\Delta_k^n}\, d\nu_x^n \qquad \forall x \text{ off this null set}$$

In fact $T\chi_{\Delta_k^n}(x) = \int \chi_{\Delta_k^n}\, d\nu_x^m \quad \forall m > n$

Now

$$\int |\nu_x^n|_p(\Delta) \, d\mu(x) = \int \sum_{k=1}^{2^n} |W_k^n(x)|^p \, d\mu(x) = \sum_{k=1}^{2^n} \int |W_k^n(x)|^p \, d\mu(x)$$

$$\leq \|T\|^p \sum_{k=1}^{2^n} \|\chi_{\Delta_k^n}\|_p^p$$

$$= \|T\|^p \cdot 1 \qquad \forall n$$

Furthermore

$$|\nu_x^{n+1}|_p(\Delta) = \sum_{k=1}^{2^{n+1}} |W_k^{n+1}(x)|^p \geq |\nu_x^n|_p(\Delta) \qquad \forall x$$

Thus $\lim_n$ of $|\nu_x^n|_p(\Delta)$ exists finitely a.e. and the limit is a measurable function of x whose integral is $\leq \|T\|^p$

Note $\|\gamma\| \leq (|\gamma|_p(\Delta))^{1/p}$ for any measure

Let ν_x be a weak* cluster point of ν_x^n off the null set.

10/20 L_p

$$\nu_x^n = \sum_{k=1}^{2^n} b_{k,x}^n$$

$$b_{k,x}^n = w_k^n(x)\,\delta_{\tau_{n,k}} \quad , \quad \tau_{n,k} \in \Delta_k^n$$

$$\nu_x^m(\Delta_k^n) = \nu_x^n(\Delta_k^n) \qquad \forall m \geq n$$

$$\int |\nu_x^n|_p(\Delta)\, d\mu(x) < \|T\|^p < \infty$$

$$|\nu_x^n|_p(\Delta) \leq |\nu_x^{n+1}|_p(\Delta)$$

So $\sup_n |\nu_x^n|_p(\Delta) < \infty$ a.e. Let $\mu(\Delta_0) = 0$, Δ_0 Borel set $\sup < \infty$ on Δ_0. Now $\|\nu_x^n\| \leq |\nu_x^n|_p(\Delta)^{1/p}$ always, so $\sup \|\nu_x^n\| < \infty$. Let ν_x be a w^*-cluster point of ν_x^n. The cluster point is unique (Recall! $\nu_x^m(\Delta_k^n) = \nu_x^n(\Delta_k^n)$ $\forall m \geq n$, so $\nu_x(\Delta_k^n) = \nu_x^n(\Delta_k^n)$ - the seq (Δ_k^n) is dense) So ν_x is the w^* limit of ν_x^n. It now follows that $x \longrightarrow \nu_x$ is $\mathcal{B}$- W^*-measurable. As before

$$\int |\nu_x^m|_p (\Delta_k^n)\, d\mu(x) \leq \|T\|^p \mu(\Delta_k^n) \quad (m \geq n)$$

By lower semi continuity

$$\int |\nu_x|_p (\Delta_k^n)\, d\mu(x) \leq \limsup \int |\nu_x^m|_p (\Delta_k^n)\, d\mu(x)$$

$$\leq \|T\|^p \mu(\Delta_k^n)$$

Now it follows by monotone class lemma that

$$\int |\nu_x|_p (B)\, d\mu(x) \leq \|T\| \mu(B)$$

for all $B \in \mathcal{B}$

Now $x \longrightarrow \nu_x$ satisfies

(i) $\mathcal{B}$-$\mathcal{W}^k$ measurable

(ii) $\int |\nu_x|_p (B)\, d\mu(x) \leq \|T\|^p \mu(B) \quad \forall$ Borel B

Thus we induce an operator S by

$$Sf(x) = \int f \, d\nu_x$$

$\|S\| \leq \|T\|$. In fact

$$\|S\| \le \sup_{\mu(B)>0} \left(\int |\nu_x|_p (B) \, d\mu(x) \, \frac{1}{\mu(B)} \right)^{1/p} \le \|T\|$$

But of course $S = T$ because they agree on $\mathcal{X}_{\Delta_k^n}$. Thus

$$\|T\| = \sup_{\mu(B)>0} \left(\int |\nu_x|_p (B) \, d\mu(x) \, \frac{1}{\mu(B)} \right)^{1/p}$$

$\boxtimes$

Note ν_x is purely atomic, so

$$\nu_x = \sum a_n(x) \, \delta_{c_n(x)}$$

Definition: $\mathcal{U}^* = \bigcap_{\gamma} (\mathcal{W}^*)_\gamma$ $\quad$ γ is a finite regular measure on $\mathcal{W}^*$

$\uparrow$ completion

Proposition: $\exists$ functions $h_n : \mathcal{M}(\Delta) \to \Delta$, $b_n : \mathcal{M}(\Delta) \to \mathbb{R}$

and $\varphi : \mathcal{M}(\Delta) \to \mathcal{M}(\Delta)$ s.t.

(i) h_n is $\mathcal{U}^*$-$\mathcal{B}$ measurable

(ii) b_n is $\mathcal{U}^*$-$\mathcal{B}_{\mathbb{R}}$ measurable

(iii) φ is $\mathcal{U}^*-\mathcal{W}^*$ measurable

(iv) $|b_n| \geq |b_{n+1}|$

(v) $h_n(\tau) \neq h_m(\tau)$ if $n \neq m$, $\tau \in \mathcal{M}(\Delta)$

(vi) $\varphi(\tau) \in \mathcal{M}_c(\Delta) = \{\nu : \nu\{x\} = 0 \ \forall x\}$

(vii) $$\lambda = \underbrace{\sum_{n=1}^{\infty} b_n(\lambda)\, \delta_{h_n(\lambda)}}_{\text{atomic}} + \underbrace{\varphi(\lambda)}_{\text{non-atomic}} \qquad \forall \lambda.$$

Given an operator T on $L_p(\Delta, \mathcal{B})$, we know it comes from $x \to \nu_x$ where ν_x is purely atomic a.e. Using proposition

$$\nu_x = \sum_{n=1}^{\infty} b_n(x)\, \delta_{h_n(x)}$$

(ν_x is purely atomic $\Rightarrow \varphi(x) = 0$) The thing to check now is that $b_n(x) = b_n(\nu_x)$ and $h_n(x) = h_n(\nu_x)$ is measurable

10/22 L_p

If $T : L_p(\Delta) \to L_p(\Delta)$, T comes from $x \to \nu_x$, $x \to \nu(x)$

$$\nu_x = \sum_{n=1}^{\infty} h_n(\nu_x)\, \delta_{b_n(\nu_x)} + \cancel{\varphi(\nu_x)}^{\,0}$$

purely atomic

$\overline{\varphi}_n(x) := h_n(\nu_x)$
$\overline{\Phi}_n(x) := b_n(\nu_x)$

$\overline{\Phi}_n$ is μ-measurable. Define a measure γ on $\mathcal{M}(\Delta)$ by

$$\gamma(C) = \mu(\nu^{-1}(C))$$

Let B be a Borel set. $b_n^{-1}(B)$ is in the completion of W^* with respect to γ. So there are two sets $C_1, C_2 \in W^*$ so that $C_1 \subset b_n^{-1}(B) \subset C_2$ and such that $\gamma(C_2 \setminus C_1) = 0$. So

$$\nu^{-1}(C_1) \subset \nu^{-1}(b_n^{-1}(B)) \subset \nu^{-1}(C_2) \text{ and } \mu(\nu^{-1}(C_2) \setminus \nu^{-1}(C_1)) = 0$$

Borel sets

$\therefore \nu^{-1}(b_n^{-1}(B))$ is a Lebesgue measurable set. Thus $\overline{\Phi}_n$ is Lebesgue measurable (similarly for $\overline{\varphi}_n$). Finally, to tidy everything up, can take functions φ_n and Φ_n equivalent a.e. to $\bar{\varphi}_n$ and $\bar{\Phi}_n$ which are Borel measurable. Then

$$Tf(x) = \sum_{n=1}^{\infty} \varphi_n(x)\, f(\Phi_n(X))$$

$\boxtimes$

The φ_n, Φ_n satisfy (1) $\sum_{n=1}^{\infty} |\varphi_n(x)|^p < \infty$ a.e.

(2) $$\sum_{n=1}^{\infty} \int_{\Phi_n^{-1}(B)} |\varphi_n(x)|^p \, d\mu(x) \leq \|T\|^p \mu(B)$$

$$\left(\int_{\Delta} |\nu_x|(B) \, d\mu(x) \leq \|T\|^p \mu(B) \right)$$

(3) $Tf(x) = \sum \varphi_n(x)\, f(\Phi_n(x))$

(4) $|\varphi_n(x)| \geq |\varphi_{n+1}(x)|$

(5) $\Phi_n(x) \neq \Phi_m(x)$ a.e.

In fact, if C is compact metric and μ is a probability measure on C, $(\mathfrak{X}, \Sigma, \lambda)$ is a measure space, λ prob measure on $\mathfrak{X}$, then given $T: L_p(C, \mathcal{B}, \mu) \to L_p(\mathfrak{X}, \Sigma, \lambda)$ $(0<p<1)$,

$$Tf(x) = \sum \varphi_n(x) f(\Phi_n(x)) \text{ a.e.}$$

where φ_n is Σ-$\mathcal{B}_{\mathbb{R}}$ measurable

Φ_n is $\Sigma \to \mathcal{B}$ measurable

<u>PROPOSITION</u>: $L_p(I^2, \mathcal{B}, m^2)$ does not project onto $L_p(I^2, \mathcal{V}, m^2)$ $\quad 0<p<1$

$\uparrow$ σ(vertical strips)

Stone Spaces

X set

Σ algebra of subsets of X $\quad \mu$ measure on Σ

Then there is a compact Hausdorff space S which has a base of clopen sets, and there is a Boolean isomorphism of Σ onto the algebra of clopen subsets of X such that μ can be transferred to a measure on the Stone space S

Remark: (1) If Σ is countable, S is metrizable (countable base for topology)

(2) If $\Sigma_1 \supset \Sigma_2$, Σ_2 subalgebra of Σ_1, $\exists$ continuous map from S_{Σ_1} onto S_{Σ_2}.

Fact: (Lusin's Lemma) Suppose K_1, K_2 compact metric, μ Borel measure on K_1, $\Theta: K_1 \longrightarrow K_2$ Borel measurable. Then $\forall \varepsilon > 0$ $\exists$ closed subset $A \subset K_1$ s.t. $\Theta|_A$ is cont. & $\mu(A^c) < \varepsilon$

10/27 L_p

$$\begin{tikzcd} C(K_A) \arrow[r, dashed, "T_h"] & C(K_B) \end{tikzcd}$$

$$\begin{tikzcd}
\text{Stone spaces} & K_A & K_B \\
\text{Boolean alg} & \mathcal{A} \arrow[u, "\tau_A"] \arrow[r, "h"] & \mathcal{B} \arrow[u, "\tau_B"]
\end{tikzcd}$$

τ_A and τ_B send the elements to corresponding clopen sets.

If h is a Boolean homomorphism, $\exists$ linear operator $T_h : C(K_A) \to C(K_B)$

such that $T_h(\chi_{\tau_A(c)}) = \chi_{\tau_B(h(c))}$ $\quad \forall c \in \mathcal{A}$

In our situation $\mathcal{A}$ is a subalgebra of $\mathcal{B}$ and h

is the injection of $\mathcal{A}$ into $\mathcal{B}$.

$$\begin{tikzcd}
C(K_a) \arrow[r, "T"] & C(K_{\mathcal{B}}) \\
\mathcal{A} \arrow[u, "\tau_{\mathcal{A}}"] \arrow[r, "i"'] & \mathcal{B} \arrow[u, "\tau_B"]
\end{tikzcd}$$

$\tau_a(A) = \chi_{C_A}$

clopen set $\sim A$

(1) $T \geq 0$ (T takes characteristic functions of clopens to clopens)

(2) Suppose C_1, C_2 are disjoint clopens in K_a. Chase the diagram to see that T takes the characteristic functions of these to characteristic functions of disjoint clopens.

(3) T is an isometry because it is an isometry on the span of the characteristic functions of the clopens

$$C(K_a)^* \xleftarrow{T^*} C(K_B)^*$$

Claim: $\exists$ cont. function $\varphi : K_B \xrightarrow{\text{onto}} K_a$

Let $\nu_x = T^* \delta_x$. Show point mass. Suppose C_1 and C_2 are disjoint clopens in K_a. $T^*\delta_x(\chi_{C_1}) = \delta_x(T\chi_{C_1})$,

$T^*\delta_x(\chi_{C_2}) = \delta_x(T\chi_{C_2})$ are not both non-zero

T is an isometry and positive, so $T^*\delta_x = a(x)\,\delta_{\varphi(x)}$

$$T^*\delta_x(\mathbb{1}) = 1$$

$$\therefore a(x) = 1$$

Note $x \to \varphi(x)$ is continuous

φ is onto: Suppose $y \in K_A$. $\delta_y \in C(K_A)^*$

Since T^* is onto, $\exists$ something in $C(K_B)^*$ which maps onto δ_y. $\{z : T^* z = \delta_y, \|z\| \le 1\}$ is w^*-closed and convex and so it has an extreme point $\bar{z}$. Claim $\bar{z}$ is extreme in $\mathrm{Ball}(C(K_B)^*)$. Thus $\bar{z} = \delta_x$ for some x (+ point mass since T^* positive)

$\mathcal{A}$ is a subalgebra of $\mathcal{B}$ translates to: the algebra $\{\varphi^{-1}(A) : A \text{ Borel set in } K_a\}$ is a subalgebra of the Borel sets of B. For suppose C is clopen in K_a. Pick x any element of K_B

$$(T^*\delta_x)\chi_C = \delta_{\varphi(x)}\chi_C = \begin{cases} 1 & x \in \varphi^{-1}(C) \\ 0 & \notin \end{cases}$$

$$\|$$

$$\delta_x(T\chi_C) = \delta_x \chi_{\hat{C}} = \begin{cases} 1 & x \in \hat{C} \\ 0 & \notin \end{cases} \quad \therefore \hat{C} = \varphi^{-1}(C)$$

(Remark which should have come earlier)

T_λ comes from $x \to \nu_x$ $\qquad$ ν_x is unique a.e.

If T also comes from $x \longrightarrow \gamma_x$, then for any clopen in Δ (the Cantor set)

$$(\gamma_x - \nu_x)\chi_c = 0 \quad \text{a.e.}$$

So throw out countably many null sets.

THEOREM: This is no projection of $L_p(I^2, \mathcal{B}, m^2)$ onto $L_p(I^2, \mathcal{V}, m^2)$ $\quad$ $\mathcal{V}$ vertical subalgebra

Proof. Let $\mathcal{B}_0$ be a countable subalgebra of $\mathcal{B}$ which generates $\mathcal{B}$. Let $\mathcal{V}_0$ be the corresponding thing for $\mathcal{V}$, chosen so that $\mathcal{V}_0 \subset \mathcal{B}_0$

$$
\begin{tikzcd}
K_{\mathcal{V}_0} & K_{\mathcal{B}_0} \arrow[l, "\varphi"'] \\
C(K_{\mathcal{V}_0}) \arrow[r, "T"] & C(K_{\mathcal{B}_0}) \\
\mathcal{V}_0 \arrow[u, "I_{\mathcal{V}_0}"'] \arrow[r, "i"] & \mathcal{B}_0 \arrow[u, "I_{\mathcal{B}_0}"']
\end{tikzcd}
$$

Let μ be the measure on $K_{\mathcal{B}_0}$ induced by m^2

Let ρ be the measure on $K_{\mathcal{V}_0}$ $\rho(B) = \mu(\varphi^{-1}(B))$

There is a natural map $S_\varphi : C(K_{\mathcal{V}_0}) \to C(K_{\mathcal{B}_0})$

$$S_\varphi(f) = f \circ \varphi$$

$$S_\varphi : L_p(K_{\mathcal{V}_0}, \mathcal{V}_0, \rho) \to L_p(K_{\mathcal{B}_0}, \mathcal{B}_0, \mu)$$

is an isometry

10/29 L_p

$$\begin{tikzcd}
C(K_{\mathcal{A}}) \arrow[r, "T"] & C(K_{\mathcal{B}}) \\
K_{\mathcal{A}} & K_{\mathcal{B}} \arrow[l, "\phi"'] \\
\mathcal{A} \arrow[u] \arrow[r, "i"] & \mathcal{B} \arrow[u]
\end{tikzcd}$$

$$T_i = S_\varphi \qquad S_\varphi f = f \circ \varphi$$

$$S_\varphi : C(K_{\mathcal{A}}) \longrightarrow C(K_{\mathcal{B}}) \quad \text{Isometry}$$

Let μ be a probability measure on $K_{\mathcal{B}}$

Induced measure $\varphi^* \mu$ on $K_{\mathcal{A}}$ $\quad \varphi^*\mu(A) = \mu(\varphi^{-1}(A))$

S_φ is an isometry of $L_p(K_{\mathcal{A}}, \underbrace{\hat{\mathcal{B}}}_{\text{Borel}}, \varphi^*\mu)$ into $L_p(K_{\mathcal{B}}, \underbrace{\hat{\mathcal{B}}}_{\text{Borel}}, \mu)$

$S_\varphi(L_p(K_{\mathcal{A}}, \text{Borel}, \varphi^*\mu))$ = all L_p functions in $L_p(K_{\mathcal{B}}, \text{Borel}, \mu)$ which are measurable w.r.t. the algebra $\{\varphi^{-1}(B) : B \text{ Borel in } K_{\mathcal{A}}\}$

PROPOSITION: In this set up assume there is an operator T from $L_p(K_{\mathcal{B}}, \text{Borel}, \mu)$ into $L_p(K_a, \text{Borel}, \varphi^*\mu)$ such that $TS_\varphi = \text{identity}$ on $L_p(K_a, \text{Borel}, \varphi^*\mu)$

Then there are $\varepsilon > 0$ and $\overset{\text{Borel}}{\theta}: K_a \to K_{\mathcal{B}}$ such that

(i) for all Borel B, $\mu(B) > \varepsilon(\varphi^*\mu)(\theta^{-1}B)$

(ii) $\varphi\theta = \text{id}$ on K_a

Proof later

THEOREM: In this set up, suppose $\exists$ $T: L_p(K_{\mathcal{B}}, \text{Borel}, \mu)$ into $L_p(K_a, \text{Borel}, \varphi^*\mu)$ such that $TS_\varphi = \text{id}$. Then $\exists$ A of positive measure in $K_{\mathcal{B}}$ such that if B is a Borel subset of A, then

$B = A \cap \varphi^{-1}(C)$ $\quad C$ Borel in K_a

Proof. $\theta: K_a \to K_{\mathcal{B}}$ $\quad \varphi\theta = \text{id}$ $\quad$ (from proposition)

K_a and $K_{\mathcal{B}}$ are compact metric spaces, so $\exists$ seq (E_n) of closed sets in K_a s.t.

$\Theta|_{E_n}$ is cont. $\forall n$

$\varphi^*\mu\left(\bigcup_n E_n\right) = 1$

Let $A = \Theta(\bigcup E_n)$ This is F_σ set $\therefore$ Borel

(i) $\mu(A) \geq \varepsilon(\varphi^*\mu)(\varphi^{-1}A) \geq \varepsilon(\varphi^*\mu)(\bigcup E_n) = \varepsilon$

(ii) Suppose B is Borel in A

$$B = \varphi^{-1}\Big(\underbrace{\Theta^{-1}(B)}_{C}\Big) \cap A$$

$\boxed{/\!/}$

~~Suppose δ_x is point mass at some point in $K_{\mathcal{B}}$~~

~~Suppose not~~

Proof of proposition: The form of T is

$$Tf(x) = \sum \varphi_n(x)\, f\,\Phi_n(x)$$

where (i) $\Phi_n : K_a \longrightarrow K_{\mathcal{B}}$

(ii) $\varphi_n : K_a \longrightarrow \mathbb{R}$

(iii) $|\varphi_n| \geq |\varphi_{n+1}|$

(iv) $\Phi_n \neq \Phi_m \ , \ n \neq m$

(v) $\displaystyle\sum \int_{\Phi_n^{-1}(B)} |\varphi_n(x)|^p \, d(\varsigma^{*}\mu)(x) \leq \|T\|^p \mu(B) \quad \forall \text{Borel } B$

10/31 L_p

(Proof-cont.) Suppose $Tf(x) = \varphi(x) f\,\Phi(x)$. Follows easily

In general $Tf(x) = \sum \varphi_n(x) f\,\Phi_n(x)$

$$\therefore f(x) = TS_\varphi f(x) = \sum \varphi_n(x) f(\varphi(\Phi_n(x)))$$

So a.e. in $\delta_x = \sum \varphi_n(x)\, \delta_{\varphi(\Phi_n(x))}$ (uniqueness of measures)

$\left(If(x) = \int f\, d\nu_x \text{ so } x \to \nu_x \text{ represents } I \right)$

pt. mass

Define a new operator $V: L_p(K_{\mathcal{B}}, \text{Borel}, \mu) \longrightarrow L_p(K_{\mathcal{A}}, \text{Borel}, \varphi^* \mu)$

by

$$Vf(x) = \sum \varphi_n(x) \chi_n(x) f\,\Phi_n(x)$$

$$\text{where } \chi_n(x) = \begin{cases} 1 & \varphi\Phi_n(x) = x \\ 0 & \varphi\Phi_n(x) \neq x \end{cases}$$

Then V is well-defined. $VS_\varphi = I$

$$Vf(x) = \sum \hat{\varphi}_n(x) f(\Phi_n(x))$$

if $\hat{\varphi}_n(x) \neq 0$, $\varphi\Phi_n(x) = x$

Take $\Phi_1 =: \Theta$

$$\sum \int_{\Phi_n^{-1}(B)} |\hat{\varphi}_n(x)|^p \, d(\varphi^* \mu)(x) \leq \|V\|^p \mu(B)$$

$\forall B$ Borel in $K_{\mathcal{B}}$. Suppose $B = \varphi^{-1}(C)$. Then

$$\sum \int_C |\hat{\varphi}_n(x)|^p \, d(\varphi^* \mu)(x) \leq \|V\|^p \underbrace{\mu \varphi^{-1}(C)}_{\varphi^* \mu(C)}$$

Then $\forall C$ Borel in K_a

$$\int_C \sum |\hat{\varphi}_n(x)|^p \, d(\varphi^* \mu)(x) \leq \|V\|^p \varphi^* \mu(C)$$

$$\therefore \sum |\hat{\varphi}_n(x)|^p \leq \|V\|^p \quad \text{a.e. } (\varphi^* \mu)$$

Now

$$1 \underset{\uparrow\, VS_\varphi = I}{\leq} \sum |\hat{\varphi}_n(x)| = \sum |\hat{\varphi}_n(x)|^p |\hat{\varphi}_n(x)|^{1-p}$$

$$\leq |\hat{\varphi}_1(x)|^{1-p} \left(\sum |\hat{\varphi}_n(x)|^p \right)$$

$$\le |\hat{\varphi}_1(x)|^{1-p} \ \|V\|^p$$

$$\therefore \ |\hat{\varphi}_1(x)| \ge \left(\frac{1}{\|V\|^p}\right)^{1/(1-p)}$$

$$\left(\frac{1}{\|V\|^{p^2}}\right)^{\frac{1}{1-p}} \varphi^*\mu(\Phi_1^{-1}(B)) \le \int_{\Phi_1^{-1}(B)} |\hat{\varphi}_1(x)|^p \, d\varphi^*\mu(x) \le \|V\|^p \mu(B)$$

$$= \left(\frac{1}{\|V\|^{p^2}}\right)^{\frac{p}{1-p}} \mu(\varphi^{-1}\Phi_1^{-1}(B)) = K \ \varphi^*\mu(\Theta^{-1}B)$$

$$\therefore \ \mu(B) \ge \frac{1}{\|V\|^p}\left(\frac{1}{\|V\|^p}\right)^{p/1-p} \varphi^*\mu(\Theta^{-1}B)$$

$\blacksquare$

$\mathcal{B}$

Suppose we have $L_p(K, \text{Borel}, \mu)$. Let $\mathcal{A}$ be a subalgebra of $\mathcal{B}$

Then there exists a projection of $L_p(K, \mathcal{B})$ onto $L_p(K, \mathcal{A})$ iff

$\exists$ Borel set $A \subset \mathcal{B}$, $\mu(A) > 0$, s.t.

① If $C \subset A$ is Borel, then $\exists\ B \in \mathcal{A}$ s.t. $B \cap A = C$

② For all $B \subset \mathcal{A}$, $\mu(B \cap A) \geq \varepsilon \mu(B)$

THEOREM: Let $T : L_p(K_1, \mathcal{B}, \mu) \longrightarrow L_p(K_2, \mathcal{B}, \nu)$ be a continuous non-zero operator. Then T preserves a copy of L_p

Open problem: If E is a complemented subspace of $L_p[0,1]$, is E isomorphic to $L_p[0,1]$.

11/3 Lp

Chapter 4 (K-spaces)

Example: ℓ_1 is not a K-space

Definition: A short exact sequence is a sequence

$$0 \longrightarrow \mathfrak{X} \xrightarrow{j} \mathcal{Y} \xrightarrow{q} Z \longrightarrow 0$$

(Image = kernal)

so j 1-1, q quotient map, kernal q = Image j

isomorphism (F-spaces)

Suppose given a locally bounded space Z with dense subspace Z_0.

$F: Z_0 \to \mathbb{R}$ is a function satisfying

(i) $F(\alpha z) = \alpha F(z)$

(ii) $|F(z_1 + z_2) - F(z_1) - F(z_2)| < K(\|z_1\| + \|z_2\|)$

quasi-norm

Let $Y_0 = \mathbb{R} \times Z_0$ with quasi-norm $|||(r,z)||| = |r - F(z)| + \|z\|$

We can construct a short exact sequence

$$0 \longrightarrow \mathbb{R} \xrightarrow{j} Y_0 \xrightarrow{q} Z_0 \longrightarrow 0$$

$$j(r) = (r,0) \qquad q(r,z) = z$$

Now $|||\, j(r) \,||| = |||\, (r,0) \,||| = |r|$ so j is isometry. Also

$$|||\, (r,z) \,||| = |r - F(z)| + \|z\| \geq \|z\| \text{ and } |||\, (F(z), z) \,||| = \|z\|$$

Thus q is a quotient map (norm agrees with quotient map)

LEMMA: If $F : Z_0 \to \mathbb{R}$ has been defined then there is a natural extension of the short exact sequence

$$0 \longrightarrow \mathbb{R} \xrightarrow{j} Y_0 \xrightarrow{q} Z_0 \longrightarrow 0$$

to

completions

$$0 \longrightarrow \mathbb{R} \xrightarrow{j} \hat{Y}_0 \xrightarrow{q} \hat{Z}_0 \longrightarrow 0$$

where j is still an isometry and q is still a quotient map

Proof j is as before and $j(\mathbb{R})$ is closed in $\hat{Y}_0$. Also $\operatorname{Im} j \subset \ker q$. Suppose $w \in \ker q$. There exists sequence $(r_n, z_n) \longrightarrow w$ with $z_n \in Z_0$ and $q(r_n, z_n) = z_n \longrightarrow 0$ since $q(w) = 0$. Now consider the sequence $(F(z_n), z_n)$

$$|||(F(z_n), z_n)||| = \|z_n\| \longrightarrow 0$$

so $\underbrace{(r_n, z_n) - (F(z_n), z_n)}_{= (r_n - F(z_n), 0) \in \operatorname{Im} j \ \forall n} \longrightarrow w$

$$\therefore w \in \operatorname{im} j$$

Finally, q is onto. Pick $z \in \hat{Z}_0$. Choose seq $z_n \in Z_0$, $z_n \to z$

~~Then consider~~ $\cancel{((F(z_n), z_n))}$

$$\cancel{|||(F(z_n), z_n) - (F(z_m), z_m)||| = |||(F(z_n) - F(z_m), z_n - z_m)|||}$$

$$\cancel{(F(z_n - z), z_n - z) \longrightarrow 0}$$

Assume $\sum \|z_n - z_{n+1}\| < \infty$. Pick $y_n \in Y_0$ s.t. $q(y_n) = z_n - z_{n+1}$ and $\|y_n\| \leq 2\|z_n - z_{n+1}\|$. Then $\left(\sum_{i=1}^{n} y_i\right)$ is Cauchy and $q(\text{limit}) = z$

11/5 L_p

Fact: $|x \log|x|| \le \frac{1}{e}$ on $[-1,1]$

Lemma: If $x, y \in \mathbb{R}$

$$|(x+y)\log|x+y| - x\log|x| - y\log|y|| \le K(|x|+|y|)$$

Proof: Suppose $x, y > 0$. Show

$$\frac{x\log x}{x+y} + \frac{y\log y}{x+y} - \frac{(x+y)\log(x+y)}{x+y} < K$$

i.e. $$\frac{x}{x+y}\log x + \frac{y}{x+y}\log y - \log(x+y)$$

$$= \frac{x}{x+y}\left(\log x - \log(x+y)\right) + \frac{y}{x+y}\left(\log y - \log(x+y)\right)$$

$$= \frac{x}{x+y}\log\frac{x}{x+y} + \frac{y}{x+y}\log\frac{y}{x+y} \le \frac{2}{e}$$

Consider now the case $x>0, y<0$, $x+y>0$. Apply above to

$-y>0$ and $x+y>0$

$$|-y\log|y| + (x+y)\log|x+y| - x\log|x|| \le 2/e(|-y|+|x+y|)$$

$$\leq \frac{4}{e}(|x|+|y|)$$

Other cases are similar

Define F on finitely non-zero sequences in ℓ_1 by

$$F((x_1,\dots,x_n)) = \sum_{i=1}^{n} x_i \log|x_i| - \left(\sum_{i=1}^{n} x_i\right)\log\left|\sum_{i=1}^{n} x_i\right|$$

Check homogeneous. $x=(x_1,\dots,x_n)$ $y=(y_1,\dots,y_n)$

$$F(x+y)-F(x)-F(y) \leq \frac{8}{e}(\|x\|+\|y\|)$$

For each i

$$\left| x_i\log|x_i| + y_i\log|y_i| - (x_i+y_i)\log|x_i+y_i|\right| \leq \frac{4}{e}(|x_i|+|y_i|)$$

$$\left| \sum x_i\log|x_i| + \sum y_i\log|y_i| - \sum (x_i+y_i)\log|x_i+y_i|\right|$$

$$\leq \frac{4}{e}\left(\sum|x_i| + \sum|y_i|\right)$$

$$\leq \frac{4}{e}(\|x\|+\|y\|)$$

For other part we have from our lemma

$$\left(\sum x_i\right)\log\left|\sum x_i\right| + \sum y_i \log\left|\sum y_i\right| - \sum(x_i+y_i)\log\left|\sum(x_i+y_i)\right|$$

$$\le \frac{4}{e} \left| \sum x_i \right| + \left| \sum y_i \right|$$

$$\le \frac{4}{e} \left(\|x\| + \|y\| \right)$$

$\therefore$ claim established

We have exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow Y \longrightarrow \ell_1 \longrightarrow 0$$

$$Y/\mathbb{R} \cong \ell_1 \quad \text{(Open Mapping Thm)}$$

Want to show $j(\mathbb{R})$ is uncomplemented in Y, which is to say that every continuous linear functional on Y vanishes on $j(\mathbb{R})$, (For if $f(r) \neq 0$, then $Y = \mathbb{R} \oplus \ker f$) i.e. which is to say

$(1,0) \in \operatorname{conv} U$ for every nbhd U of 0 in Y_0

Fix n. For $1 \le k \le n$ let

$$z_n^k = \frac{1}{\log n}\left(\frac{-1}{n-1}, \ldots, 1, \ldots, \frac{-1}{n-1} \right)$$

$\uparrow$ k^{th} coordinate

Then $(1,0) = \frac{1}{n} \sum_{k=1}^{n} (1, z_n^k)$

$$|||\,(1, z_n^k)\,||| = |\,1 - F(z_n^k)\,| + \underbrace{\|z_n^k\|}_{= \frac{2}{\log n} \to 0}$$

$$F(z_n^k) = \frac{1}{\log n} F\Big(-\frac{1}{n-1}, \dots, \underbrace{1}_{k^{th}\text{ spot}}, \dots -\frac{1}{n-1}\Big)$$

$$= \frac{1}{\log n}\left((n-1)\left(\frac{-1}{n-1}\right)\log\frac{1}{n-1}\right)$$

$$= \frac{\log(n-1)}{\log n} \qquad (\text{for each } k)$$

$$\longrightarrow 1 \text{ uniformly in } k$$

$\therefore\ (1,0) \in \operatorname{conv} U$ for every nbhd U of 0 in Y_0

11/7 L^p

LEMMA: Let $\mathfrak{X}$ be a metrizable TVS and suppose $\mathcal{Y}$ is a closed subspace of $\mathfrak{X}$ s.t. both $\mathcal{Y}$ and $\mathfrak{X}/\mathcal{Y}$ are locally bounded. Then $\mathfrak{X}$ is locally bounded.

Fact: Let $\mathfrak{X}$ be metrizable. Let $A \subset \mathfrak{X}$.

① If A is unbounded, then $\exists$ sequence $(\alpha_n) \subset \mathbb{R}$, $\alpha_n \to 0$ and a sequence $(x_n) \subset A$ s.t. $(\alpha_n x_n)$ is unbounded

② If A is bounded, then whenever $(x_n) \subset A$ and $\alpha_n \to 0$, $\alpha_n \in \mathbb{R}$, then $\alpha_n x_n \to 0$

Proof of lemma: Let $\mathcal{U}$ be a nbhd of 0 in $\mathfrak{X}$ s.t. $\mathcal{U}$ is balanced and

(i) $\pi(\mathcal{U})$ is bounded in $\mathfrak{X}/\mathcal{Y}$

(ii) $\mathcal{U} \cap \mathcal{Y}$ is bounded

$[-1,1]V \subset V$

Let V be a nbhd of 0 with $V + V \subset \mathcal{U}$ and V balanced.

Claim: V is bounded. Suppose not. Pick $\alpha_n \to 0$, $x_n \in V$ s.t.

$(\alpha_n x_n)$ is unbounded. However, $\alpha_n \pi(x_n) = \pi(\alpha_n x_n) \to 0$ so there exists $(y_n) \subset \mathcal{Y}$ s.t. $\alpha_n x_n + y_n \to 0$ in $\mathcal{X}$. Thus $\alpha_n x_n + y_n \in V$ for all $n \geq N$ and so $y_n \in V - V \subset \mathcal{U}$. Hence (y_n) is bounded $\Rightarrow$ $\alpha_n x_n = (\alpha_n x_n - y_n) + y_n$ is bounded $\lightning$

$\boxtimes$

Suppose $\mathcal{Z}$ is locally bounded and $F: \mathcal{Z} \to \mathbb{R}$ is quasilinear. Define $\mathbb{R} \oplus_F \mathcal{Z} = \{(r,z) : r \in \mathbb{R}, z \in \mathcal{Z}\}$ with

$$|||(r,z)||| = |r - F(z)| + \|z\|$$

What does it mean for $\mathbb{R}$ to be complemented in $\mathcal{Z}$? Suppose $P: \mathbb{R} \oplus_F \mathcal{Z} \to \mathbb{R}$ is a projection.

$$P(r,z) = P(r,0) + P(0,z)$$

$$= (r,0) + T(z)$$

$$\|P(F(z), z)\| \leq K \, |||(F(z), z)||| = K\|z\|$$

$$\uparrow = ((F(z), z) + T(z))$$

$$\therefore \; | F(z) + \hat{T}(z) | \le K\|z\|$$

$$(\; T(z) = (\hat{T}(z), 0))$$

$\therefore$ F can be approximated by a linear map

[Show directly that for the F constructed on ℓ_1, $\nexists \; T : (\ell_1)_0 \to \mathbb{R}$ s.t

$$| T(z) - F(z) | \le K\|z\|$$

Hint : look on basis elements and sum of 1^{st} n basis elements]

Definition: A K-space Z satisfies following : $\exists$ of exact seq

$$0 \to \mathbb{R} \xrightarrow{j} Y \xrightarrow{q} Z \to 0$$

$\Rightarrow j(\mathbb{R})$ complemented in Y

Theorem: Z is a K-space iff whenever Z_0 is a dense subspace of Z and $F : Z_0 \to \mathbb{R}$ is quasi-linear, then $\exists \; T : Z_0 \to \mathbb{R}$ linear s.t. $| T(z) - F(z) | \le K\|z\| \;\; \forall z \in Z_0$.

Proof ($\Rightarrow$) Suppose $F : Z_0 \to \mathbb{R}$. Consider

$$0 \longrightarrow \mathbb{R} \xrightarrow{j} \mathbb{R} \oplus_F Z_0 \xrightarrow{q} Z_0 \longrightarrow 0$$

Complete this to get

$$0 \longrightarrow \mathbb{R} \xrightarrow{j} \widehat{\mathbb{R} \oplus_F Z_0} \xrightarrow{q} Z \longrightarrow 0$$

Z K-space $\Rightarrow$ $\exists$ projection $\widehat{\mathbb{R} \oplus_F Z_0} \longrightarrow j(\mathbb{R})$. Now restrict projection to $\mathbb{R} \oplus_F Z_0$ and use previous result.

Now suppose quasi-linear functions can be approximated. We show Z is a K-space. Given

$$0 \longrightarrow \mathbb{R} \xrightarrow{j} Y \xrightarrow{q} Z \longrightarrow 0$$

By open mapping theorem, $\exists\, u : Z \longrightarrow Y$ s.t.

(i) $\|u(z)\| \le K_1 \|z\|$

(ii) $u(\alpha z) = \alpha u(z)$

(iii) $qu(z) = z$

Let $v : Z \longrightarrow Y$ be linear (discontinuous) s.t. $qv(z) = z$. Then $u(z) - v(z) \in \ker q$. Let $F(z) = j^{-1}(u(z) - v(z))$. Homogeneous ✓

$$F(z_1+z_2) = j^{-1}\left(u(z_1+z_2) - v(z_1+z_2)\right)$$

$$= j^{-1}\left(u(z_1+z_2) - v(z_1) - v(z_2)\right)$$

$$\therefore \left|F(z_1+z_2) - F(z_1) - F(z_2)\right| = \left|j^{-1}\left(u(z_1+z_2) - u(z_1) - u(z_2)\right)\right|$$

$$\leq \|j^{-1}\| \; \|u(z_1+z_2) - u(z_1) - u(z_2)\|$$

$$\leq \|j^{-1}\| \; K_2\left(\|z_1\| + \|z_2\|\right)$$

By assumption $\exists$ linear $T : Z \longrightarrow \mathbb{R}$ such that

$$\left|j^{-1}\left(u(z) - v(z)\right) - T(z)\right| \leq K_3 \|z\|$$

For $y \in Y$ let $Q(y) := v(q(y)) - jT(q(y))$. Then $q(Qy) = q(v(q(y)))$ $= q(y)$. From this, Q is a projection

11/10 Lp

$$0 \longrightarrow \mathbb{R} \xrightarrow{j} Y \xrightarrow{q} Z \longrightarrow 0$$

$u : Z \longrightarrow Y$ $\qquad$ $u(\alpha z) = \alpha u(z)$

$\qquad\qquad\qquad\qquad \|u(z)\| \leq k\|z\|$

$\qquad\qquad\qquad\qquad qu = \mathrm{id}$

$V : Z \longrightarrow Y$ linear $\qquad qv = \mathrm{id}$

$$F(z) = j^{-1}(u(z) - v(z))$$

Jon

Larry

To show Q is continuous:

$$\left| j^{-1}(u(z) - v(z)) + T(z) \right| \leq K_3\|z\|$$

$$\Rightarrow \| u(z) - v(z) + jT(z) \| \leq \|j\| K_3 \|z\|$$

$$\Rightarrow \| u(qy) - vq(y) + jTq(y) \| \leq \|j\| K_3 \|q(y)\| \leq \|j\| K_3 \|q\| \|y\|$$

$$\Rightarrow \| vq(y) - jTq(y) \| \leq \cancel{\|j\| K_3 \|q\| \|y\|} \leq K_4 \|y\|$$

since $\|u(qy)\| \leq K'\|y\|$ $\qquad\qquad = Q(y)$

Finally $(I - Q)(y) = y - vq(y) + jTq(y)$. This shows $I - Q$

is projection onto $\ker q$ $\qquad \blacksquare$

Remark: No special property of $\mathbb{R}$ was used

Definition: (Z, X) splits if every short exact sequence

$$0 \longrightarrow X \xrightarrow{j} Y \xrightarrow{q} Z \longrightarrow 0$$

(Z, X locally bdd, complete) has the property that $j(X)$ is completed in Y. Alternatively, for every quasi-linear map $F: Z_0 \longrightarrow X$ $\exists$ linear $T: Z_0 \longrightarrow X$ s.t. $\|F(z) - T(z)\| \leq K\|z\|$

Z_0 dense in Z

(Replace $\mathbb{R} \oplus_F Z$ by $X \oplus_F Z$)

Theorem: Let $1 \geq q > p$. Let X be a q-convex F-space. Then (ℓ_p, X) splits.

Claim: Suppose $\ell_p^n \longrightarrow X$ is quasi-linear with constant K

$$\|F(x+y) - F(x) - F(y)\| \leq K(\|x\| + \|y\|)$$

If $x = \sum \alpha_i e_i \in \ell_p^n$. Then $\left\| F\left(\sum_{i=1}^n \alpha_i e_i\right) - \sum_{i=1}^n \alpha_i F(e_i) \right\|$

$$\leq K\left(\sum_{i=1}^n \left(\frac{2}{i}\right)^{q/p}\right)^{1/q} \left\|\sum_{i=1}^n \alpha_i e_i\right\|_p$$

Proof by induction: $n=1$ ok ✓

Suppose true for $n-1$

$$X = \alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_j e_j + \dots + \alpha_k e_k + \dots + \alpha_n e_n$$

$j \neq k$ indices s.t. $|\alpha_j|^p + |\alpha_k|^p \leq \frac{2}{n} \sum_{i=1}^{n} |\alpha_i|^p$. If no such j, k

look at 2-element subsets. and add up all possibilities to get ⊂

Consider span $e_1, \dots, e_{j-1}, \alpha_j e_j + \alpha_k e_k, \dots e_n \cong \ell_p^{n-1}$

isometric

normalize

By induction

$$\| F\left(\sum_{i=1}^{n} \alpha_i e_i\right) - \sum_{\substack{i=1 \\ i \neq j,k}}^{n} \alpha_i F(e_i) - F(\alpha_j e_j + \alpha_k e_k) \|$$

(*)

$$\leq K\left(\sum_{i=1}^{n-1} \left(\frac{2}{i}\right)^{q/p}\right)^{1/q} \| \sum_{i=1}^{n} \alpha_i e_i \|_p$$

Now $\| F(\alpha_j e_j + \alpha_k e_k) - \alpha_j F(e_j) - \alpha_k F(e_k) \| \leq K(\|\alpha_j\| + \|\alpha_k\|)$

But $|\alpha_j| + |\alpha_k| \leq (|\alpha_j|^p + |\alpha_k|^p)^{1/p}$

~~So (*) is~~ $\leq K(|\alpha_j|^p +$

$$\left\| F\left(\sum \alpha_i e_i \right) - \sum \alpha_i F(e_i) \right\|$$

$$\leq \left\| F\left(\sum_{i=1}^{n} \alpha_i e_i \right) - \sum_{\substack{i=1 \\ i \neq j,k}}^{n} \alpha_i F(e_i) - \alpha_j F(e_j) - \alpha_k F(e_k) \right.$$

$$\left. + F(\alpha_j e_j + \alpha_k e_k) - F(\alpha_j e_j + \alpha_k e_k) \right\|$$

11/12 L_p

$$0 \longrightarrow X \longrightarrow Y \xrightarrow{q} \ell_p \longrightarrow 0$$

(Use map on particular dense subset to get right inverse for any dense subset)

<u>Facts</u>: Let U, V be sym., balanced bounded sets in a TVS

a) $U+V$ is bounded

b) If U, V are p-convex, so is $U+V$, i.e. if $x, y \in X$, $\alpha^p + \beta^p = 1$ $\Longrightarrow \alpha x + \beta y \in X$ ("p-convex")

Lemma: Suppose (E, X) splits. Suppose

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "j"] & Y \arrow[r, "q"] & Z \arrow[r] & 0 \\ & & & E \arrow[u, "S"] \arrow[ul, dashed, "\tilde{S}"] & \end{tikzcd}$$

$S: E \to Z$ linear operator. Then S lifts to $\tilde{S}: E \to Y$ s.t. $q\tilde{S} = S$

Proof. Define $G = \{(e,y) \in E \oplus Y : Se = qy\}$. Note $(0, j(X)] \subset G$. Define $\pi: G \to E$ by $\pi(e,y) = e$. Then $\ker \pi = \{(0,y) : qy = S0 = 0$, i.e. $y \in j(X)\}$. Identify $\ker \pi$ with $j(x)$

We now have

$$0 \longrightarrow X \xrightarrow{\hat{j}} G \xrightarrow[\text{onto}]{\pi} E \longrightarrow 0$$

(E, X) splits, so $\exists$ linear $T : E \longrightarrow G$ s.t. $\pi T(e) = e$.

$$T(e) = (e, y_e)$$

Let $\tilde{S}(e) = y_e$. Then $q\tilde{S}(e) = Se$ $\blacksquare$

<u>THEOREM</u>: Suppose

$$0 \longrightarrow X \xrightarrow{j} Y \xrightarrow{q} Z \longrightarrow 0$$

is a short exact sequence, X q-convex, Z p-convex, $q > p$. Then Y has a p-convex quasi-norm

Proof

$$\begin{tikzcd} & & \ell_p(I) \arrow[d, "S"', "\text{onto}"] & \\ 0 \arrow[r] & X \arrow[r, "j"] & Y \arrow[r, "q"] & Z \arrow[r] & 0 \end{tikzcd}$$

Let I be an index set $S : \ell_p(I)$ an onto continuous linear map

Let $\{z_i : i \in I\}$ be dense in unit sphere of Z. Define $S : \ell_p(I) \longrightarrow Z$

by $S(f) = \sum_{i \in I} f(i) z_i$

(unconditionally convergent)

Lift S to $\tilde{S} : \ell_p(I) \longrightarrow Y$. Let $U = \tilde{S}(B_{\ell_p(I)})$, $V = j(B_X)$

Then V is q, and hence p-convex. U is p-convex. Show

$U+V$ is bounded and absorbing. Bounded clear. Given $y \in Y$

Then $q(y) \in \lambda S(B_{\ell_p(I)})$. Since $q\tilde{S} = S$ and S is onto, then

$y \in \lambda U + j(X)$. Say $y = \lambda u + v$ $\quad v \in j(X) \subset \ell V$

$\therefore y \in \max(\lambda, \ell)(U+V)$

$U+V$ is bounded, absorbing, p-convex, so by category

$\overline{U+V}$ is bounded, p-convex and a nbhd of the origin.

$\boxtimes$

Corollary: For $0<p<1$, L_p is a K-space

Proof. If $0 \longrightarrow \mathbb{R} \longrightarrow Y \longrightarrow L_p \longrightarrow 0$ is short exact sequence, then Y is p-convex. Now use previous lifting theorem.

<u>K spaces</u>	<u>Not K space</u>
$L_p \quad 0 \le p < 1$	ℓ_1
L_0	L_1 (since contains ℓ_1 as complemented subspace)
$L_p, \ell_p \quad p > 1$	

Theorem: If X and Z are B-convex Banach spaces

$$\inf_{\varepsilon_i = \pm 1} \sup_{\|x_i\| \le 1} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| = o(n)$$

then if $0 \to X \to Y \to Z \to 0$ is a short exact seq, then Y is locally convex

Problem: Which of these are K-spaces?

$c_0, \ \ell_\infty, \ C(K), \ H_p \ (p<1)$

11/14 L_p

$$0 \longrightarrow X \longrightarrow Y \xrightarrow{q} \ell_p \longrightarrow 0$$

define u, v as before

$F = j^{-1}(u-v)$

$F|_{Z_0}$

approx by L to get right inverse for q on Z_0

$\exists$ cont. linear inverse for q which extends to all of ℓ_p

$qT = \mathrm{id}$ on ℓ_p $\quad P_y := Tq(y)$

Compact convex sets with no extreme points

Take cone

Definition: Suppose $A \subset B$ in metric space.

$$d(A,B) := \sup_{b \in B} \inf_{a \in A} d(a,b)$$

Lemma: Suppose (A_n) is an increasing sequence of compact subsets of a metric space and $\sum d(A_n, A_{n+1}) < \infty$. Then $\bigcup A_n$ is pre-compact.

Proof. Given $\varepsilon > 0$ Choose k s.t. $\sum_{n=k}^{\infty} d(A_n, A_{n+1}) < \varepsilon/2$

Let (z_i) be an $\varepsilon/2$ net for $\bigcup_{j=1}^{k} A_j$. Then (z_i) is ε net for $\bigcup A_i$

Lemma: Suppose $(x_1, \dots, x_n) \in$ F-space X. Suppose F_i, $i = 1, \dots, n$ $\varepsilon > 0$

are finite sets s.t.

(i) $x_i \in \operatorname{conv} F_i$) not necessary

ii) if $y \in \operatorname{conv} F_i$, then $\exists\, \alpha \in [0,1]$ s.t. $d(y, \alpha x_i) < \varepsilon/n$

Then if $z \in \operatorname{conv} \bigcup_{i=1}^{n} F_i$, ~~$\exists$ numbers $\alpha_1, \dots, \alpha_n$ in $(0,1)$ s.t.~~

$$\cancel{z = \sum_i \alpha_i y_i}$$

$\exists\, w \in \operatorname{conv} \{x_1, \dots, x_n\} \cup \{0\}$ s.t. $d(z, w) < \varepsilon$

Proof. Suppose $z = \sum_i \sum_{j=1}^{|F_i|} \alpha_j^i x_j^i$ $\quad x_j^i \in F_i$

$$\sum_i \sum_j \alpha_j^i = 1$$

$$= \sum_i \left(\frac{\sum_{j=1}^{|F_i|} \alpha_j^i x_j^i}{\sum_{j=1}^{|F_i|} \alpha_j^i} \right) \sum_{j=1}^{|F_i|} \alpha_j^i$$

$$= \sum_{i=1}^{n} \Big(\sum_{j=1}^{|F_i|} \alpha_j^i \Big) q_i \qquad q_i \in \operatorname{conv} F_i$$

By (ii) $\exists\ \beta_i \in [0,1]$ s.t. $d(q_i, \beta_i x_i) < \varepsilon/n$

$i = 1, \dots, n$

Then

$$d\Big(\sum_{i=1}^{n} \Big(\sum_{j=1}^{|F_i|} \alpha_j^i \Big) q_i \,,\ \underbrace{\sum_{i=1}^{n} \Big(\sum_{j=1}^{|F_i|} \alpha_j^i \Big) \beta_i x_i}_{\omega} \Big) < n \cdot \varepsilon/n$$

$(\,|tx| \le |x|,\ |t| \le 1\,)$

$$\sum_{i=1}^{n} \sum_{j=1}^{|F_i|} \alpha_j^i \beta_i \le 1$$

11/17 L_p

<u>Definition</u>: A point x in an F-space $\mathfrak{X}$ is called a <u>needlepoint</u> of $\mathfrak{X}$ if for every $\varepsilon > 0$ there is a finite set $F \subset \mathfrak{X}$ s.t.

(1) $x \in \operatorname{conv} F$

(2) $|y| < \varepsilon$ if $y \in F$

(3) If $z \in \operatorname{conv} F$, $\exists\, \alpha \in [0,1]$ s.t. $|z - \alpha x| < \varepsilon$

<u>Definition</u>: $\mathfrak{X}$ is a needlepoint space if every point of $\mathfrak{X}$ is a needlepoint of $\mathfrak{X}$

<u>Proposition</u>: Suppose $\mathfrak{X}$ is a needlepoint space. Then we can construct a bush such that if $D_n = \operatorname{conv}\{0, x_k^i : i \le n\}$ ← level of bush

with $d(D_n, D_{n+1}) \le \varepsilon_n$ (where $\sum \varepsilon_n < \infty$)

Proof: Suppose we have already constructed the 1^{st} n rows.

Given ε_n, for each i, $1 \le i \le k$, there is a finite set F_i in $\mathfrak{X}$

with

↑ #elements in n^{th} row

i) $|y| \leq \varepsilon_n \quad \forall y \in F_i$

ii) $x_i^n \in \text{conv } F_i$

iii) if $y \in \text{conv } F_i$, $\exists\, \alpha_i \in [0,1]$ s.t.

$$|y - \alpha_i x_i^n| < \varepsilon_n / k$$

Let $F = \bigcup_i F_i$ and $y \in \text{conv } F_i$. Then by last time's lemma,

$\exists\, z \in \text{conv}\{0, x_1^n, \dots, x_k^n\}$ s.t. $|z - y| < \varepsilon_n$. F is $(n+1)^{st}$ row

Note: $\bigcup_n D_n$ is pre-compact

<u>Definition:</u> $x \in \mathfrak{X}$ is an approximate needlepoint of $\mathfrak{X}$ if

for every $\varepsilon > 0$ $\exists$ finite set $F \subset \mathfrak{X}$ s.t.

(i) for some $y \in \text{conv } F$, $|x - y| < \varepsilon$

(ii) $|z| < \varepsilon \quad \forall z \in F$

(iii) if $z \in \text{conv } F$, then $\exists\, \alpha \in [0,1]$ s.t. $|z - \alpha x| < \varepsilon$.

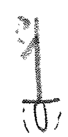
Facts about approximate needlepoints

(1) The set of approx. needlepoints of $\mathfrak{X}$ is closed in $\mathfrak{X}$

(2) Suppose x is an approx. needlepoint and $T : \mathfrak{X} \to \mathfrak{X}$ is a continuous linear operator. Then Tx is an approx. needlepoint.

THEOREM: $\mathbb{1}$ is an approx. needlepoint of L_p, $0 \le p < 1$.
(Proof later)

Corollary: All constants are approx. needlepoints. If $|f| \le M$ a.e.
Then $f = T(\mathbb{1})$
then $x \xrightarrow{T} f \cdot x$ is cont. on L_p, $0 \le p < 1$, so f is an approx. needlepoint by (2). So L_p is an approx. needlepoint space by (1)

$\mathbb{1} =$  $= \delta_{\text{needle}}$

<u>Proposition</u>: An approximate needlepoint space is a needlepoint space.

Proof: Given $\varepsilon > 0$, $x \in \mathfrak{X}$. $\exists$ finite set $F_1 \subset \mathfrak{X}$ s.t.

(i) - (iii) satisfied for x, ε. Let $x_1 \in \operatorname{conv} F$, $|x - x_1| < \varepsilon$.

Find a finite set $F_2 \subset X$ satisfying (i) - (iii) for $x - x_1$ and $\varepsilon/2$.

(Choose $x_2 \in \operatorname{conv} F_2$ s.t. $|x - x_1 - x_2| < \varepsilon/2$.) Keep this up.

Having defined $x_1, \dots, x_n$ find a finite set $F_{n+1} \subset X$ satisfying

(i) - (iii) for $x - x_1 - \dots - x_n$, $\varepsilon/2^{n+1}$. Also $\exists\, x_{n+1} \in \operatorname{conv} F_{n+1}$

s.t. $|x - x_1 - x_2 - \dots - x_{n+1}| < \varepsilon/2^{n+1}$

Claims: (1) $\sum_{i=1}^{\infty} x_i = x$

? (2) $|x_{n+1}| \leq |x - \dots - x_{n+1}| + |x - \dots - x_n| \leq \frac{\varepsilon}{2^{n+1}} + \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2^n}$

(3) $x_1 = \sum_j \alpha_j z_j^1 \quad z_j^1 \in F_1 \quad |z_j^1| \leq \varepsilon/2 \;\; \forall j$

Let $x^j = z_j^1 + x_2 + x_3 + \dots$ Then $|x^j| \leq \varepsilon/2 + \varepsilon = 3\varepsilon/2$. Also

$x \in \operatorname{conv}\{x^j\}$

11/19 L_φ

Suppose $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$

φ cont. $\varphi(0) = 0$

$\varphi \nearrow$

φ subadditive

φ concave

$\varphi(x)/x \to 0$ as $x \to \infty$

$$L_\varphi = \left\{ f : \int \varphi(|f|(x))\,dx < \infty \right\}$$

F-norm

Lemma: Suppose $f \in L_1$. Then $\int_0^1 \varphi(|f|(x))\,dx \le \varphi\left(\int_0^1 |f(x)|\,dx\right)$

Proof. Suppose $x_0 = \int_0^1 |f(x)|\,dx$. Let $Lx = ax + b$, $Lx_0 = \varphi(x_0)$

Then $\varphi(x) \le Lx$. Now

$$\int_0^1 \varphi(|f|) \le \int_0^1 (a|f| + b) = ax_0 + b = \varphi(x_0)$$

$$= \varphi\left(\int_0^1 |f|\right)$$

$\blacksquare$

We work on $\Omega = \prod_{i=1}^{\infty} [0,1]$ with product Lebesgue measure. Given f measurable on $[0,1]$, define $(S_i f)$ on Ω by $S_i f(x) = f(x_i)$

The $S_i f$ are independent, having the same distribution as f.

$$\operatorname{Var} f = \int f^2 - \left(\int f\right)^2$$
$$= \int \left(f - \int f\right)^2$$

If $f_1, \ldots, f_n$ are independent, then

$$\operatorname{Var} \sum f_i = \sum \operatorname{Var} f_i$$

$$\operatorname{Var}(\alpha f) = \alpha^2 \operatorname{Var}(f)$$

Lemma: Suppose $f \in L_2[0,1]$. Consider $S_i(f)$ $1 \le i \le n$ and scalers α_i, $\alpha_i \ge 0$, $\sum \alpha_i \le 1$ $\quad \int f = 1$. Write

$$N(f) = \int \varphi(|f|)$$

Then $N\left(\sum_{i=1}^{n} \alpha_i S_i f - \sum_{i=1}^{n} \alpha_i\right) \le \varphi\left(\left(\alpha \operatorname{Var}(f)\right)^{1/2}\right)$ $\quad (\alpha = \max \alpha_i)$

Proof. $N\left(\sum \alpha_i S_i(f) - \sum \alpha_i\right)$

$$\le \varphi\left(\left\| \sum (\alpha_i S_i f - \alpha_i) \right\|_1\right)$$

$$\le \varphi\left(\left\| \sum (\alpha_i S_i f - \alpha_i) \right\|_2\right)$$

$$= \varphi\left(\left(\sum \mathrm{Var}\,(\alpha_i S_i(f))\right)^{1/2}\right)$$

$$= \varphi\left(\left(\sum_{i=1}^{n} \alpha_i^2 \,\mathrm{Var}\, f\right)^{1/2}\right)$$

$$\leq \varphi\left(\left(\alpha \sum_{i=1}^{n} \alpha_i \,\mathrm{Var}\, f\right)^{1/2}\right)$$

$$\leq \varphi\left(\left(\alpha \,\mathrm{Var}\, f\right)^{1/2}\right)$$

$\boxtimes$

Note It isn't hard to get functions f in L^{∞} s.t

i) $f \geq 0$

ii) $\int f = 1$

iii) $N(f)$ is small

Take $f = \dfrac{\chi_E}{m(E)}$. $N(f) = \varphi\left(\frac{1}{m(E)}\right) m(E) \longrightarrow 0$ as $m(E) \to 0$

Suppose $\delta > 0$, $b > 0$, $N(f) < \delta/k$ where $1/k < b$, $\sum_{i=1}^{n} \alpha_i \leq 1$.

Assume $\alpha_i > b$. Then

$$\frac{n}{k} \leq nb \leq \sum_{i=1}^{n} \alpha_i \leq 1$$

$$\therefore \; n < k$$

Thus $N\left(\sum_{i=1}^{n} \alpha_i S_i(f)\right) \leq K \cdot \frac{\delta}{K} = \delta$

Suppose $0 < a < b \leq 1$, $\delta > 0$. The interval $[a,b]$ is a <u>δ-divergent zone</u> for f ($f \geq 0$, $\int f = 1$, $f \in L_\infty$) if

① $N\left(\sum_{i=1}^{n} \alpha_i S_i f - \sum \alpha_i\right) \leq \delta$ if $\alpha_i \leq a \ \forall i$, $\sum \alpha_i \leq 1$

② $N\left(\sum_{i=1}^{n} \alpha_i S_i f\right) < \delta$ if $\alpha_i \geq b \ \forall i$, $\sum \alpha_i \leq 1$

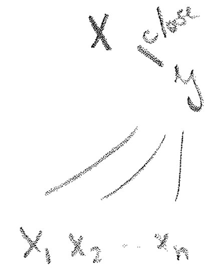
Claim: Given $\delta > 0$, $K \in \mathbb{N}$, we can find $f_1, \ldots, f_K$ ($f_i \geq 0$, $\int f_i = 1$, $f_i \in L_\infty$) s.t. f_i has a δ-divergent zone $[a_i, b_i]$ and $[a_i, b_i] \cap [a_j, b_j] = \phi$ $i \neq j$

Take f s.t. $N(f_1) < \frac{\delta}{j}$ from note

Let b_1 be any number < 1. Choose a_1 s.t. $\varphi\left((a_1 \operatorname{Var} f_1)^{1/2}\right) < \delta$

Let $b_2 < a_1$. Choose f_2 etc.

11/21 L_p



$y \in \text{conv}(x_1, \dots, x_n)$

Then $x \in \text{conv}\left(x_1 + (x-y),\ x_2 + (x-y), \dots, x_n + (x-y) \right)$

$\therefore$ approx needlepoint is a needlepoint

Given $k, \delta > 0$ $\exists f_i$, $1 \le i \le k$ $\int f_i = 1$ $f_i \ge 0$ $f_i \in L_\infty$

s.t. $[a_i, b_i]$ is a δ-divergent zone for f_i, $[a_i, b_i] \cap [a_j, b_j] = \phi$ $i \ne j$

Given $\varepsilon > 0$, choose K so that $\varphi\left(\frac{1}{K}\right) < \varepsilon/3$. Now chose $\delta = \varepsilon/3K$. Choose for this k, δ $f_1, \dots, f_k$ with pairwise disjoint δ-divergent zones as above.

$$f = \frac{1}{K} \sum_{j=1}^{k} f_j$$

$$\therefore \sum_{l=1}^{n} \alpha_i S_i(f) = \frac{1}{K} \sum_{j=1}^{k} \underbrace{\sum_{i=1}^{n} \alpha_i S_i(f_j)}_{= L_j + M_j + R_j}$$

where $L_j = \sum \alpha_i S_i(f_j) \quad \alpha_i < a_j$

$$M_j = \sum \alpha_i S_i(f_j) \quad a_j \le \alpha_i \le b_j$$

$$R_j = \sum \alpha_i S_i(f_j) \quad b_j < \alpha_i$$

$N(R_j) < \delta$ (by δ-divergent zones)

$N(L_j - c_j) < \delta$ where $c_j = \sum \alpha_i \quad \alpha_i < a_j$

$$\int M_j = \sum \alpha_i \quad \alpha_i \in [a_j, b_j]$$

$$\therefore \int \sum_{j=1}^{k} M_j = \sum_{j=1}^{k} \sum \alpha_i \quad \alpha_i \in [a_j, b_j]$$

$$\le \underline{1}$$

$$\therefore \int \frac{1}{k} \sum_{j=1}^{k} M_j \le 1/k$$

Hence

$$N\left(\frac{1}{k} \sum_{j=1}^{k} M_j\right) \le \varphi(1/k) < \varepsilon/3$$

$$N\left(\frac{1}{k} \sum_{j=1}^{k} R_j\right) \le k \cdot \delta = \varepsilon/3$$

$$N\left(\frac{1}{k} \sum_{j=1}^{k} (L_j - c_j)\right) \le k \cdot \delta = \varepsilon/3 \quad \left(1/k \left(\sum_{j=1}^{k} c_j \right) \in [0,1] \right)$$

$$N\Big(\frac{1}{k}\sum_{j=1}^{k} L_j - \underbrace{\frac{1}{k}\sum_{j=1}^{k} c_j}_{\alpha}\Big) < \varepsilon/3$$

$$\therefore \quad N\Big(\sum_{i=1}^{n} \alpha_i S_i(\xi) - \alpha \mathbb{1}\Big) < \varepsilon$$

$\boxtimes$

Definition: A compact convex set X is ε-generated for $\varepsilon > 0$

if $X = \operatorname{conv}\left(\bigcup_{i=1}^{n} X_i\right)$, $X_i \subset X$ where $x \leq \varepsilon \quad \forall x \in X_i \quad \forall i$

X_i compact convex $\forall i$

$= \left\{ \sum \alpha_i x_i : x_i \in X_i \right\}$ since X_i convex

Fact. Suppose X is ε-generated $\forall \varepsilon > 0$. Then $\operatorname{ext} X \subset \{0\}$.

Proof. Suppose x is extreme. Then if $\varepsilon > 0$

$$x = \sum_{i=1}^{n} \alpha_i x_i \qquad x_i \in X_i$$

$$\Rightarrow x = x_i \ \forall i \quad \text{since } x \text{ extreme}$$

$$\Rightarrow |x| < \varepsilon$$

ε arbitrary $\Rightarrow x = 0$

Suppose we have constructed non-trivial $\hat{X}$ compact convex, ε-generated $\forall \varepsilon > 0$. Then $\underbrace{\mathrm{conv}(\hat{X} \cup -\hat{X})}_{\text{compact, convex}}$ has no extreme points

The X-bush constructed earlier has these properties.

Roberts used needlepoints to construct a non-trivial twisted sum of B and $\mathbb{R}$, for some B Banach space

$$0 \longrightarrow \mathbb{R} \xrightarrow{j} Y \xrightarrow{i} B \longrightarrow 0$$

($j(\mathbb{R})$ is uncomplemented)

11/24 L_p

<u>Definition</u>: Suppose X is an F-space. X has the Hahn-Banach approximation property if the weak closure of every proper closed subspace is proper.

Suppose $0 \to \mathbb{R} \xrightarrow{j} Y \to X \to 0$ is a short exact sequence with X a Banach space. Then Y has HBAP

Proof. Suppose M is proper and closed in Y. Assume first that $j(\mathbb{R}) \subset M$. The map

$$Y/j(\mathbb{R}) \longrightarrow Y/M$$

is continuous and onto. $Y/j(\mathbb{R})$ is Banach, so Y/M is Banach. Since dual of a Banach space separates points, the weak closure of M is M.

In general we can assume that Y is not locally convex.

Then $M + j(\mathbb{R}) \neq Y$ (If not, $j(\mathbb{R})$ is complemented)

$M + j(\mathbb{R})$ is proper closed and by previous argument the weak closure of $M \subset$ weak closure of $M + j(\mathbb{R}) = M + j(\mathbb{R})$.

$\boxtimes$

In ℓ_p, $0 < p \leq \partial$, let ℓ_p^σ = finitely non-zero sequences in ℓ_p. Define $F : \ell_p^\sigma \longrightarrow \ell_p^\sigma$ by

$$F(x)_i = x_i \log \left(\frac{|x_i|}{\|x\|} \right) \qquad x_i \neq 0$$

$$= 0 \qquad x_i = 0$$

Note $F(\lambda x) = \lambda F(x)$

i^{th} coordinate of $F(x+y) - F(x) - F(y)$ is

$$(x+y)_i \log \frac{|(x+y)_i|}{\|x+y\|} - x_i \log \frac{|x_i|}{\|x\|} - y_i \log \frac{|y_i|}{\|y\|}$$

$$= (x+y)_i \log |(x+y)_i| - x_i \log |x_i| - y_i \log |y_i| \qquad (A)$$

$$- (x+y)_i \log \|x+y\| + x_i \log \|x\| + y_i \log \|y\| \qquad (B)$$

$$A \leq \frac{4}{e}\left(|x_i| + |y_i|\right), \text{ so } \|A\| \leq K_p\left(\|x\| + \|y\|\right)$$

$$\|B \text{ term}\| = \left\| x_i \log \frac{\|x\|}{\|x+y\|} + y_i \log \frac{\|y\|}{\|x+y\|} \right\|$$

$$\cancel{\leq K_p\left(\log\left(\frac{\|x\|}{\|x+y\|}\right)\|x\| + \log\left(\frac{\|y\|}{\|x+y\|}\right)\|y\|\right)}$$

$$\frac{\|B \text{ term}\|}{\|x+y\|} \leq K_p\left(\frac{\left|\log\left(\frac{\|x\|}{\|x+y\|}\right)\right|}{\|x+y\|}\|x\| + \frac{\left|\log\left(\frac{\|y\|}{\|x+y\|}\right)\right|}{\|x+y\|}\|y\|\right)$$

$$\leq K_p'$$

$$\therefore \|B \text{ term}\| \leq K_p'\|x+y\| \leq \hat{K}_p\left(\|x\| + \|y\|\right)$$

$\therefore$ F is quasi-linear.

$$0 \longrightarrow \ell_p^\sigma \longrightarrow \ell_p^\sigma \oplus_F \ell_p^\sigma \longrightarrow \ell_p^\sigma \longrightarrow 0$$

When we complete we get

$$0 \longrightarrow \ell_p \longrightarrow Z_p \longrightarrow \ell_p \longrightarrow 0$$

Type

For $0 < p \leq 2$, a quasi-normed space X is of type p if there exists a constant K_p s.t. for all finite sequences $x_1, \dots, x_n$ in X

$$\| \int_0^1 \sum_{i=1}^n r_i(t) x_i \|^p \leq K_p \sum_{i=1}^n \|x_i\|^p$$

$$\|$$

$$\sum_{\varepsilon_i = \pm 1} \frac{\| \sum_{i=1}^n \varepsilon_i x \|^p}{2^n}$$

Hilbert spaces are of type 2

Fact: For $0 < p \leq 2$, ℓ_p is of type p

Theorem: $0 \to \ell_p \to Z_p \to \ell_p \to 0$ does not split

Proof. We need only show that there does not exist an $H : \ell_p^\sigma \to \ell_p^\sigma$ linear and a $K > 0$ s.t.

$$\| H(x) - F(x) \| \leq K \|x\| \quad \forall x \in \ell_p^\sigma$$

Suppose $\exists$ such H, K. $F(e_i) = 0 \Rightarrow \|H(e_i)\| \leq K$

Given n let $x_n(t) = \sum_{i=1}^{n} r_i(t) e_i$

$$F(x_n(t))_i = -r_i(t) \cdot \log(n^{1/p})$$

11/26 L_p

(Proof cont.) $\quad x_n(t) = \sum_{i=1}^{n} r_i(t) e_i$

$$F(x_n(t)) = -\sum_{i=1}^{n} r_i(t) \log(n^{1/p})^{e_i} = -\frac{1}{p} \sum_{i=1}^{n} r_i(t)(\log n) e_i$$

So $\|F(x_n(t))\| = \frac{1}{p} \log n \; n^{1/p}$

Now $\left(\int_0^1 \|F(x_n(t))\|^p dt\right)^{1/p} = \frac{1}{p} \log n (n^{1/p})$

$$\int_0^1 \|H(x_n(t))\|^p dt = \int_0^1 \|\sum_{i=1}^{n} r_i(t) H(e_i)\|^p dt$$

$$\underset{\uparrow \text{type ineq}}{\leq} K_p \sum_{i=1}^{n} \|H(e_i)\|^p \leq K_p n K^p$$

$$\therefore \int_0^1 \|F(x_n(t)) - H(x_n(t))\|^p dt$$

$$\geq \left(\frac{1}{p}\right)^p (\log n)^p n - K_p n K^p \qquad (1)$$

$$= n\left(\left(\frac{1}{p}\right)^p (\log n)^p - K_p K^p\right)$$

But

$$\int_0^1 \| F(x_n(t)) - H(x_n(t)) \|^p \, dt \leq \int K^p \|x_n(t)\|^p dt \leq K^p n \quad (2)$$

(1) + (2) can't hold for large n

Remarks: For $p \leq 1$, once we know that $\|H(e_i)\| \leq K$, then H is a continuous linear operator:

$$\| H(\Sigma \alpha_i e_i) \|_p^p \leq \Sigma |\alpha_i|^p \|H(e_i)\|^p \leq K^p \Sigma |\alpha_i|^p$$

And then if $x_n = \sum_{i=1}^{n} e_i$,

$$\| H(x_n) - F(x_n) \| \leq K \|x_n\|$$

is impossible since it forces $\|F(x_n)\|$ to be small.

For $p > 1$, Z_p is a Banach space since the twisted sum of B-convex Banach spaces is a Banach space

For $p \leq 1$, Z_p is not p-convex. For suppose Z_p is p-convex

$Z_p/\ell_p \rightsquigarrow \ell_p$

$$0 \longrightarrow \ell_p \xrightarrow{i} Z_p \xrightarrow{q} \ell_p \longrightarrow 0$$

If this is p-convex, $\exists$ lift backward, i.e. right inverse for q

By Open Mapping th^m you can choose x_i in Z_p, $\|x_i\| \leq K \ \forall i$ s.t. $q(x_i) = e_i$. Define $v: \ell_p \to Z_p$ by $v(e_i) = x_i$. By p-convexity this is cont. and right inverse for q $\lightning$

Kalton: If $0 \to X \to Y \to Z \to 0$, and if X is p-convex, Z q-convex, $q > p$, then Y is p-convex.

Question: For $p > 1$, is Z_p isomorphic to a closed subspace of codimension $\underline{1}$?

12/1 L_p

Nikishin Factorization of Operators into $L_0(\Omega, \mu, \mathfrak{X})$

prob. space ↑ ↑ Banach space

Will take $\mathfrak{X} = \mathbb{R}$

Suppose $\alpha \in (0,1)$

$$J_\alpha(f) = \inf\{c \in \mathbb{R} : \mu(|f| > c) \leq \alpha\}$$

Note that the L_0-topology is generated by the family of all J_α's

Facts: (1) $J_\alpha(cf) = |c| \, J_\alpha(f)$

(2) $J_\alpha(f+g) \leq J_{\alpha/2}(f) + J_{\alpha/2}(g)$

Suppose B is a Banach space and $T: B \to L_0(\Omega, \mu)$ is continuous linear operator.

If $f \in L_1(\Omega, \mu)$, then for all $c > 0$

$$c\,\mu\{|f| > c\} \leq \int |f|$$

quasi-norm

Weak $L_1 = L^{1,\infty} = \{f \in L_0(\Omega, \mu) : \sup_{c>0} c\,\mu(|f| > c) < \infty\}$

$1/x \in L^{1,\infty}$ but not $L^1[0,1]$

"Fubini" Th^m Suppose (Ω_1,μ_1) (Ω_2,μ_2) are probability spaces and $f: \Omega_1 \times \Omega_2 \to \mathbb{R}$ is strongly measurable Suppose $\alpha,\beta,\gamma,\delta$ are in $(0,1)$. Then

$$J_\alpha\left(J_\beta f(x,y)\,d\mu_2(y)\right)d\mu_1(x) \qquad (*)$$

$$\leq J_\gamma\left(J_\delta f(x,y)\,d\mu_1(x)\right)d\mu_2(y)$$

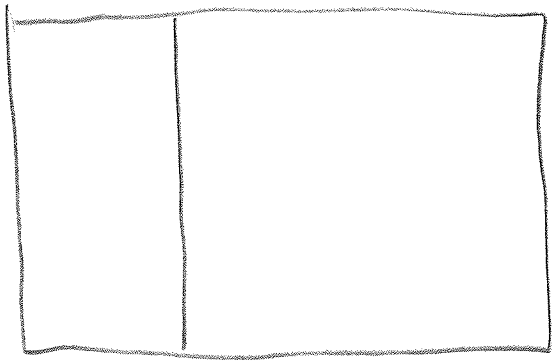
provided $\gamma+\delta \leq \alpha\beta$

(Fact: Suppose $f \in L_0(\Omega)$ $\quad c = J_\alpha(f)$. Then $\mu\{|f|>c\} \leq \alpha$)

Proof for $\Omega_1 = \Omega_2 = [0,1]$ with Lebesgue measure. Suppose $\gamma+\delta > \alpha\beta$ but $(*)$ goes other way

$$J_\alpha\left(J_\beta f(x,y)\,dm(y)\right)dm(x) > \underbrace{J_\gamma\left(J_\delta f(x,y)\,dm(x)\right)dm(y)}_{=k}$$

$\therefore m\{x : J_\beta f(x,y)\,dm(y) > k\} > \alpha$



α

For these x's, $\int_\beta f(x,y)\,dm(y) > k$

$\Rightarrow$ $|f(x,y)| > k$ on a set E^x of y-measure $> \beta$

So from Fubini's theorem,

$$m^2\{\,|f(x,y)| > k\,\} > \alpha\beta$$

Now do other order

$$\int_\gamma \left(\int_\delta f(x,y)\,dm(x)\right) dm(y) = k$$

$$\Rightarrow m\left\{\int_\delta f(x,y)\,dm(x) > k\right\} \le \gamma$$

γ

For y not in the bad set $J_\delta f(x,y) \le k$, so

$$m\{x : |f(x,y)| > k\} \le \delta$$

By Fubini $m^2\{|f(x,y)| > k\} \le \delta + \gamma$

$\boxtimes$

$R > 0$

THEOREM: Let $\varepsilon \in (0,1)$, (Ω, μ) prob. space and let $A \subset L_0(\Omega, \mu)$. If

(1) $\forall (c_n) \subset \mathbb{R}$ s.t. $\sum |c_n| \le 1$ and $\forall (f_n) \subset A$

$$\mu\{\sup |c_n f_n| > R\} \le \varepsilon$$

Then (2) $\exists$ set $\Omega_\varepsilon \subset \Omega$ with $\mu(\Omega \setminus \Omega_\varepsilon) \le \varepsilon$, $\forall c > 0$ and $\forall f \in A$, $\mu\{\omega \in \Omega_\varepsilon : |f(\omega)| > c\} \cdot c < R$. In the reverse direction, if (2) holds for R & ε, then $\forall (c_n) \subset \mathbb{R}$ with $\sum |c_n| \le 1$, $\mu\{\sup |c_n f_n| > R/\varepsilon\} \le 2\varepsilon$ $\forall (f_n) \subset A$

12/3 L_p

Proof.

(Change (2) to $\exists\ \Omega_\varepsilon$ measurable, $\mu(\Omega \setminus \Omega_\varepsilon) \le \varepsilon$ such that

$$\mu\{\omega \in \Omega_\varepsilon : |f(\omega)| > Rc\} \le 1/c$$

$\forall c > 0\ \forall f \in A$)

(1) $\Rightarrow$ (2) A measurable set B is an N-set iff

(i) $\mu(B) > 0$

(ii) $\exists f \in A$ s.t. $\mu(B)|f| > R$ a.e. on B

Let $\mathcal{B}$ be a maximal family of pairwise disjoint N-sets.

$\mathcal{B} = (B_i)_{i=1}^{\infty}$. On B_i, $\mu(B_i)|f_i| > R$, so on $\cup B_i$

$\sup_i \mu(B_i)|f_i| > R$. Let $c_i = \mu(B_i)$ $\sum_i c_i \le 1$. By (1)

$\mu(\cup B_i) \le \varepsilon$. Set $\Omega_\varepsilon = \Omega \setminus \cup B_i$. ~~Suppose $\exists D \subset \Omega_\varepsilon$,~~

~~$\mu(D) > 0$ s.t.~~ If $\mu\{\underbrace{\omega \in \Omega_\varepsilon : |f(\omega)| > Rc}_{E}\} > 1/c$

Then $\mu(E)\,|g| > R$ on E. Thus E is an N-set so $\mathcal{B} \cup \{E\}$ is strictly larger than maximal $\mathcal{B}$ $\lightning$.

$(2) \Rightarrow (1)$ Suppose $\sum |c_n| \leq 1$, $(f_n) \subset A$. Let

$$B_n = \{\omega \in \Omega_\varepsilon : |c_n|\,|f_n(\omega)| > \frac{R}{\varepsilon}\} = \{\omega \in \Omega_\varepsilon : |f_n(\omega)| > \frac{R}{|c_n|\varepsilon}\}$$

If (2) holds, $\mu(B_n) \leq |c_n|\varepsilon$, so $\mu(\cup B_n) \leq \sum |c_n| \varepsilon \leq \varepsilon$

Off of $\cup B_n$, $\sup |c_n f_n| \leq R/\varepsilon$ on Ω_ε

$$\therefore \mu\left(\sup |c_n f_n| > R/\varepsilon\right) \leq \underset{\cup B_n}{\underset{\uparrow}{\varepsilon}} + \underset{\Omega \setminus \Omega_\varepsilon}{\underset{\uparrow}{\varepsilon}}$$

$\blacksquare$

Condition (1) can be rephrased as

$$J_\varepsilon(\sup (c_n f_n)) \leq R \sum |c_n|$$

for any summable sequence (c_n) and $\forall\, (f_n) \subset A$.

More facts about J_α

① $J_\alpha(|f|^p) = (J_\alpha f)^p \quad p>0$

② $J_\alpha f \leq \frac{1}{\alpha}\int |f|$ if $f \in L_1$ (Chebyscher's ineq)

If $f \in L^p$ (weak L_p)

$$\infty > \int |f|^p \geq \int_{|f|>c} |f|^p \geq c^p \mu(|f|>c)$$

$$L^{p,\infty} = \{ f : \sup_{c>0} \mu(|f|>c)\, c^p < \infty \}$$

↑ this is a number

Note: $f \in L^{p,\infty} \iff |f|^p \in L^{1,\infty}$

Goal

type p

$$\begin{tikzcd} X \arrow[rr, "U"] \arrow[dr, dashed, "\text{``almost''}"'] & & L_0 \\ & L^{p,\infty} \arrow[ur, dashed] & \end{tikzcd}$$

type p

$$\begin{tikzcd} X \arrow[rr, "U"] \arrow[dr, dashed] & & L_0 \\ & L^q \arrow[ur, dashed] & \end{tikzcd}$$

$q<p$

Definition: Let X be a Banach space. Let $u: X \longrightarrow L_0^{(\Omega,\mu)}$ be continuous linear operator. Say that u is almost-ε $L^{p,\infty}$ bounded ~~if $u(B_{\mathfrak{X}})$ satisfies (1) of the preceding~~ if $\{|u(x)|^p : x \in B_{\mathfrak{X}}\}$ satisfies (1) of the preceding theorem for some $R > 0$

(We'll show that if X is of type p, every u is almost-ε $L^{p,\infty}$ bounded $\forall \varepsilon > 0$)

Corollary: Suppose $u: X \longrightarrow L_0$ is a cont. linear operator and for all finite sequences $x_1, \dots, x_n$ in X,

$$J_\varepsilon\left(\sup_i |u(x_i)|\right) \le K_\varepsilon\left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p}$$

Then u is almost-ε $L^{p,\infty}$ bounded.

Proof. Take (c_i), $\sum |c_i| \le 1$. Let (x_i) finite seq in $\mathfrak{X}$. $\|x_i\| \le 1$

$$J_\varepsilon\left(\sup_i c_i |u(x_i)|^p\right) = J_\varepsilon\left(\sup_i \left(c_i^{1/p} |u(x_i)|\right)^p\right)$$

$$= \left(J_\varepsilon \left(\sup_i c_i^{1/p} |u(x_i)| \right) \right)^p$$

$$= \left(J_\varepsilon \left(\sup |u(c_i^{1/p} x_i)| \right) \right)^p$$

$$\leq K_\varepsilon^p \left(\sum \|c_i^{1/p} x_i\|^p \right)$$

$$\leq K_\varepsilon^p \sum |c_i| \leq K_\varepsilon^p$$

$\blacksquare$

(Use monotone result for J_ε $\quad J_\varepsilon(f_i) < C$

$\Rightarrow J_\varepsilon(\sup f_i) < c$)

12/5 Lp

$$U : \mathfrak{X} \longrightarrow L_0(\Omega, \mu)$$

Show $J_\varepsilon(\sup |U(x_i)|) \leq K_\varepsilon \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p}$

B-space

<u>Theorem</u>: <u>If</u> $\mathfrak{X}$ is of type p, then

$$J_\varepsilon(\sup |U(x_i)|) \leq K_\varepsilon \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p}$$

$\forall\ x_1, \dots, x_n \in \mathfrak{X}$.

<u>Lemma</u>: Suppose $\beta_1, \dots, \beta_n$ are in $\mathbb{R}$. Then

$$\sup_i |\beta_i| \leq J_\alpha\left(\sum_{i=1}^n \beta_i r_i \right) \quad \text{if } \alpha < 1/2$$

Proof. Fix i. Take $\beta = 2\alpha$

$$2|\beta_i| = J_\beta(2\beta_i r_i) = J_\beta\left(\beta_i r_i + \sum_{j\neq i} \beta_j r_j + \beta_i r_i - \sum_{j \neq i} \beta_j r_j\right)$$

$$\leq \underset{\uparrow \beta/2}{J_\alpha}\left(\beta_i r_i + \sum_{j\neq i} \beta_j r_j\right) + J_\alpha\left(\beta_i r_i - \sum_{j\neq i} \beta_j r_j\right)$$

$$= 2 J_\alpha\left(\sum_{i=1}^n \beta_i r_i \right) \qquad \blacksquare$$

Fix $\omega \in \Omega$.

Proof of th^m: Let $x_1, \dots, x_n \in \mathfrak{X}$. Take $\beta_i = u(x_i)(\omega)$

$$J_\varepsilon(\sup u(x_i)(\cdot)) \leq J_\varepsilon\left(J_{1/3} \sum u(x_i)(\omega)\, r_i(t)\, dt\right) d\mu(\omega)$$

$\uparrow$ $\alpha = 1/3$ in lemma

(Choose $\gamma + \delta \leq \varepsilon/3$)

$$\leq J_\gamma\left(J_\delta \sum_{i=1}^{n} u(x_i)(\omega)\, r_i(t)\, d\mu(\omega)\right) dt$$

$$= J_\gamma\left(J_\delta\, u\left(\sum_{i=1}^{n} x_i r_i(t)\right)(\omega)\, d\mu(\omega)\right) dt$$

J_δ homogeneous, u continuous $\longrightarrow$

$$\leq K_\delta\, J_\gamma\left(\left\| \sum_{i=1}^{n} x_i r_i(\cdot) \right\|\right)$$

$$\leq K_\delta \frac{1}{\gamma} \int \left\| \sum_{i=1}^{n} x_i r_i(\cdot) \right\|$$

$$\leq K_\delta \frac{1}{\gamma} \left(\int \left\| \sum_{i=1}^{n} x_i r_i(\cdot) \right\|^p \right)^{1/p} \quad (p \geq 1)$$

type $\longrightarrow$

$$\leq \underbrace{K_\delta \frac{1}{\gamma} K_p(\mathfrak{X})}_{K_\varepsilon} \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p}$$

$\boxed{/}$

Remark $\quad J_\alpha(f) \le K_\alpha \left(\int |f|^p \right)^{1/p} \qquad$ Chebyschev

so above proof works for $p<1$

<u>Theorem</u>: Let $\mathfrak{X}$ be of type p, $u: \mathfrak{X} \to L_0$ continuous.

Then u factors through $L^{p,\infty}$

$$\begin{tikzcd} \mathfrak{X} \arrow[rr, "u"] \arrow[dr, "gu"'] & & L_0 \\ & L^{p,\infty} \arrow[ur, "h \mapsto h/g"'] & \end{tikzcd}$$

Proof Let $A = u(B_{\mathfrak{X}})$. By the preceeding, $\exists$ a sequence of disjoint measurable sets (E_n) in Ω s.t. $\mu(\cup E_n) = 1$

$\exists$ seq (R_n) in $\mathbb{R}$ s.t. $\forall c>0$, all $f \in A$

$$c^p \mu\{\omega \in E_n : |f(\omega)| > c\} \le R_n$$

Define $\quad g = \dfrac{1}{R_n^{1/p} 2^n}$ on E_n

$$c^p \mu\left(\omega \in E_n : \frac{f}{R_n^{1/p} 2^n} > c\right)$$

$$= c^p \mu(w \in E_n : f > c R_n^{1/p} 2^n)$$

$$= \frac{c R_n^{1/p} 2^n}{(R_n^{1/p} 2^n)^p} \mu(w \in E_n : f > c R_n^{1/p} 2^n)$$

$$\leq \frac{R_n}{(R_n^{1/p} 2^n)^p} = \frac{1}{2^{np}}$$

$$\therefore c^p \mu\{w : |gf(w)| > c\} \leq \sum \frac{1}{2^{np}} < \infty$$

$\therefore T_1 : x \longmapsto g(\cdot) u(x)$ maps $\mathcal{X}$ into $L^{p,\infty}$

$T_2 : L^{p,\infty} \longrightarrow L_0 \quad T_2 f = \frac{f}{g}$ cont

$$U = T_2 T_1$$

$\boxtimes$

Corollary: Any operator from any Banach space to L_0 factors through $L^{1,\infty}$. (B-space of type 1)

Corollary. A Banach subspace of L_0 embeds in $L^{1,\infty}$ and embeds in L^p, $p<1$

$$\begin{tikzcd} X \arrow[rr, "U"] \arrow[dr, "T_1"'] & & L_0 \\ & L^{1,\infty} \arrow[ur, "T_2"'] & \end{tikzcd}$$

$U : X \to L_0$ isom.

T_1: Isomorphism into

2nd ~~fact~~: If $\|f\|_{L^{1,\infty}} < R$, then for $p<1$ $\|f\|_p < K_p \cdot R$

$$c\,\mu(|f|>c) \le R$$

$$\|f\|_p \qquad p\int_1^p x^{p-1}\mu(|f|>x)\,dx$$

$$= p\int_1^\infty x^{p-1}\frac{R}{x}\,dx = p\int_1^\infty x^{p-2}\cdot R$$

$$\int |f|^p\,d\mu = p\int_1^\infty x^{p-1}\,\mu(|f|>x)\,dx$$

$\blacksquare$

The Lifting Theorem for $L_0[0,1]$ Let F be a finite dimensional subspace of $L_0[0,1]$. If T is an operator from L_0 into the quotient space L_0/F and π is the canonical quotient map from L_0 onto L_0/F, then there is a unique linear operator $\tilde{T}: L_0 \to L_0$ such that $T = \pi \circ \tilde{T}$.

$$\begin{tikzcd}
 & L_0 \arrow[d, "\pi"] \\
L_0 \arrow[ur, "\tilde{T}"] \arrow[r, "T"'] & L_0/F
\end{tikzcd}$$

Proof. The uniqueness is easy. Suppose $T_1 : L_0 \to L_0$ is another operator satisfying $\pi \circ T_1 = T$. Then $\pi(\tilde{T} - T_1) = 0$ and so we may consider $\tilde{T} - T_1$ as an operator from L_0 into F. But the dual F^* has lots of continuous linear functionals while L_0 has a trivial dual. Thus we must have $\tilde{T} = T_1$.

Since F is a finite dimensional space, its topology can be given by a norm $\|\cdot\|$. We break the proof of the theorem into three steps.

Step 1: There exists $c > 0$ such that if x is a non-zero element of F, then $m(\operatorname{supp} x) > c$.

To see this, suppose the conclusion is false. Then there exists a sequence (x_n) in F such that $x_n \neq 0$ and $m(\operatorname{supp} x) \to 0$. Let $y_n = x_n / \|x_n\|$. Then $m(\operatorname{supp} y_n) \to 0$ and so $y_n \to 0$ in the L_0 norm. Therefore $\|y_n\| \to 0$ in F. But this is impossible since $\|y_n\| = 1$ for each n. This contradiction proves step 1.

Step 2: There exists $\delta > 0$ such that whenever $m(\operatorname{supp} f) < \delta$, then there exists a unique element $h(f)$ in the coset Tf of L_0/F such that $m(\operatorname{supp} h(f)) \leq c/3$.

Let us first show the uniqueness of $h(f)$. Suppose q is another element which satisfies the conclusion. Then $h(f)-q$ belongs to F and $m(\text{supp}(h(f)-q)) \leq 2c/3 < c$. By step 1 we must have $h(f)-q=0$, i.e. $h(f)=q$.

By the continuity of T there exists $\delta>0$ such that if $|f| < \delta$ then $|Tf| < c/3$. Now suppose $m(\text{supp } f) < \delta$. Then for any n, $|nf| < \delta$ and so $|T(nf)| < c/3$. Fix z in Tf. Then for each n there exists a w_n in F such that

$$(*) \qquad \int \frac{|nz+w_n|}{1+|nz+w_n|}\, dm < \frac{c}{3}$$

Case 1: (w_n/n) has a bounded subsequence

Since F is finite dimensional and passing to a subsequence if necessary, we may assume that (w_n/n) converges to some element w in F both in the norm of F and almost everywhere. Now from $(*)$ we see that

$$\int \frac{|z + w_n/n|}{\frac{1}{n} + |z + \frac{w_n}{n}|} \, dm < c/3$$

Since the integrand converges to 1 on the support of $z+w$, we must have $m(\text{supp}(z+w)) \leq c/3$. Set $h(\xi) = z+w$.

Case 2: Now suppose $\|w_n/n\| \to \infty$. Then $\|w_n\| \to \infty$. This time we have in $(*)$ that

$$\int \frac{\left| \frac{n}{\|w_n\|} z + \frac{w_n}{\|w_n\|} \right|}{\frac{1}{\|w_n\|} + \left| \frac{n}{\|w_n\|} z + \frac{w_n}{\|w_n\|} \right|} \, dm < c/3$$

Pass to a subsequence, if necessary, to get $w_n/\|w_n\|$ converging to some element w in F both in the norm of F and almost everywhere. Then the integrand tends almost everywhere to 1 on the support of w, and so $m(\text{supp } w) \leq c/3$. But this contradicts step 1. Hence case 2 is impossible and only case 1 can occur. This finishes step 2.

Step 3. For proving the theorem, let (A_i), $1 \le i \le n$, be a partition of $[0,1]$ with $m(A_i) < \delta$ for each i, where δ is obtained from step 2. For each f in L_0 define

$$\tilde{T} f = \sum_{i=1}^{n} h(f \chi_{A_i})$$

Then clearly $T = \pi \circ \tilde{T}$.

We must check that $\tilde{T}$ is linear and continuous. This will follow from the homogenerty and additivity of $h(\cdot)$. If $m(\operatorname{supp} f) < \delta$, then $h(\alpha f)$ and $\alpha h(f)$ both lie in $T(f)$ and both have the measure of their support $\le c/3$. By the uniqueness in step 2, we have $h(\alpha f) = \alpha h(f)$. Now suppose $m(\operatorname{supp} f) < \delta$, $m(\operatorname{supp} g) < \delta$, and $m(\operatorname{supp}(f+g)) < \delta$. Then $h(f+g) - h(f) - h(g)$ belongs to F and

$$m(\operatorname{supp}(h(f+g)-h(f)-h(g))) \leq c$$

so that by step 1, we have $h(f+g)-h(f)-h(g)=0$

It is now clear from the definition that $\tilde{T}$ is linear.

Towards showing the continuity of $\tilde{T}$, suppose f_k is supported on A with $m(A) < \delta$. We want to show that if (f_k) converges to 0 in measure, then $(h(f_k)) \to 0$ in measure. We know that $Tf_k \to 0$ in L_0/F, so there exists a sequence (w_k) in F such that $h(f_k)+w_k \to 0$ in measure. Suppose $w_n \not\to 0$. By passing to a subsequence if necessary, we may assume that $(w_n/\|w_n\|)$ converges to some non-zero element w in F. Then

$$\frac{h(f_n)}{\|w_n\|} \longrightarrow -w$$

But $m(\operatorname{supp}(h(f_n)/\|w_n\|)) \leq c/3$ while $m(\operatorname{supp}(-w)) > c$

by step 1. This contradiction shows that $w_n \to 0$, and therefore that $h(\delta_n) \to 0$.

Hence $\tilde{T}$ is continuous, which completes the proof.

We can use the lifting theorem to answer the following question.

<u>QUESTION</u>: Is L_0 isomorphic to the quotient space L_0/F when F is a non-zero finite dimensional subspace?

Suppose $T: L_0 \to L_0/F$ is an isomorphism. Lift T to an operator $\tilde{T}: L_0 \to L_0$. Then $\tilde{T}$ is also an isomorphism and $\tilde{T}(L_0)$ is a closed subspace of L_0 of codimension equal to the dimension of F. But this is impossible since L_0 has no continuous functionals (if x^* is a continuous linear functional on the finite dimensional quotient space $L_0/\tilde{T}(L_0)$, then $x^* \circ \pi$ is a continuous linear functional on L_0, where $\pi : L_0 \to \tilde{Y}(L_0)$ is the quotient map).

Hence the answer to the question is a resounding No.

<u>Definition</u>: An F-space X has L_0-structure if for every $\varepsilon > 0$, X is the finite direct sum of closed subspaces of diameter less than ε.

If there is any justice in the world, one would certainly expect L_0 to have L_0-structure. This expectation is correct. Given any $\varepsilon > 0$, partition $[0,1]$ into a finite number of intervals A_i of length less than ε, and set $X_i = \{ f \chi_{A_i} : f \in L_0 \}$. In fact, for any F-space Y, the F-space $L_0(Y)$ of all Y-valued measurable functions has L_0-structure.

OPEN PROBLEM: Characterize the F-spaces with L_0-structure.

__Definition:__ Suppose X is an F-space. For each x in F let

$$\sigma(x) = \sup\{ |rx| : r \text{ a real number}\}.$$

__Definition:__ An F-space X is locally bounded if its topology has a bounded neighborhood of 0, i.e. if there is a neighborhood U of 0 such that if V is any other neighborhood of 0, then nV contains U for some integer n.

If X is locally bounded, then its topology can be given by a quasi-norm $\|\cdot\|$ satisfying

$$\|x+y\| \leq K\left(\|x\| + \|y\|\right)$$

$$\|\alpha x\| = |\alpha|\,\|x\|.$$

Indeed, we may assume U is a symmetric bounded neighborhood of 0 and take $\|\cdot\|$ to be the guage (or Minkowski) functional of U.

<u>Generalized Lifting Theorem:</u> Let X be an F-space with L_0-structure. Let Y be an F-space and let B be a closed locally bounded subspace of Y. Let $T: X \longrightarrow Y/B$ be a linear operator. Then there is a unique lifting of T to a linear operator $\tilde{T}: X \longrightarrow Y$

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[dr, "\tilde{T}"'] & Y/B \\ & Y \arrow[u, "\pi"'] \end{tikzcd}$$

Proof. The uniqueness is easy. If $T = \pi T_1 = \pi \tilde{T}$, then $\pi(\tilde{T} - T_1) = 0$, so we may consider $\tilde{T} - T_1$ as an operator from X into B. But X has L_0-structure and B is locally bounded, so the continuity of $\tilde{T} - T_1$ implies that $\tilde{T} - T_1 = 0$.

Step 1. Take $\delta > 0$ such that the set $\{b \in B : |b| \leq \delta\}$ is bounded. Then given z in the quotient space Y/B with

$\sigma(z) \leq \delta/3$, there exists a unique x in Y satisfying $\pi x = z$ and $\sigma(x) \leq \delta/3$.

Towards verifying this claim, for each integer n pick an element x_n in Y such that $\pi x_n = z$ and $|n x_n| \leq (1+1/n)|nz|$. Then (x_n) is a Cauchy sequence. To see this, let $u_n = x_n - x_1$. Then u_n belongs to B. For $2 \leq k < n$,

$$\begin{aligned} |ku_n - ku_k| &= |k(x_n - x_k)| \\ &\leq |kx_n| + |kx_k| \\ &\leq |nx_n| + |kx_k| \\ &\leq (1+1/n)\delta/3 + (1+1/k)\delta/3 \\ &= (2+1/n+1/k)\delta/3 < \delta. \end{aligned}$$

Since $u_n - u_k$ lies in a bounded neighborhood of 0, the above inequality shows that (u_n) is Cauchy. Hence (x_n) is Cauchy, so $x_n \to x$ for some x in Y.

Now fix an integer n. If $\varepsilon > 0$, then for sufficiently large k we have

$$|nx| \le |nx_k| + |n(x_k - x)| \le |kx_k| + \varepsilon$$
$$\le (1 + 1/k)\sigma(z) + \varepsilon .$$

Hence $\sigma(x) \le \sigma(z) \le \delta/3$.

To see the uniqueness, suppose $\pi y_1 = z$ and $\sigma(y_1) \le \delta/3$. Then $\pi(y_1 - y) = 0$ and $\sigma(y_1 - y) \le 2\delta/3 < \delta$, i.e.

$$|n(y_1 - y)| < \delta \text{ for all } n.$$

But $y_1 - y$ belongs to B, so this last inequality is impossible unless $y_1 - y = 0$ since we have a bounded neighborhood of 0.

Step 2. Let H be a linear subspace of Y/B with $\sigma(x) \le \delta/3$ for each x in H. Then there exists a continuous linear operator $V : H \to Y$ such that $\pi V = Id_H$.

Suppose z is in H. Then $\sigma(z) \le \delta/3$. Define

$V(z)$ to be the unique y from step 1 satisfying $\pi y = z$ and $\sigma(y) \le \delta/3$. If α is a scaler, then $\pi(\alpha V(z)) = \alpha \pi V(z) = \alpha z$ and $\sigma(\alpha V(z)) = \sigma(V(z)) \le \delta/3$. By uniqueness, then, $V(\alpha z) = \alpha V(z)$. Additionally, $V(z_1 + z_2) - V(z_1) - V(z_2)$ is an element of B with

$$\sigma(V(z_1+z_2) - V(z_1) - V(z_2)) \le \delta.$$

But this is impossible in a locally bounded space unless $V(z_1+z_2) = V(z_1) + V(z_2)$. Hence V is linear on H.

For the continuity of V, let (z_n) be a sequence in H with $z_n \to 0$. Choose x_n in Y such that $\pi x_n = z_n$ and $|x_n| \le 2|z_n|$. Then $x_n \to 0$. Suppose the sequence $x_n - Vz_n$ does not converge to 0. If necessary, pass to a subsequence to get $|x_n - Vz_n| \ge \gamma > 0$. Since $x_n - Vz_n$ belongs to B and B is locally bounded, we can find a sufficiently large α so that

$$|\alpha(x_n - Vz_n)| \geq \delta$$

for each n. Then for sufficiently large n

$$|\alpha V z_n| \geq \delta - |\alpha x_n| \geq \delta - \delta/6 = 5\delta/6$$

and so $\sigma(Vz_n) = \sigma(\alpha V z_n) \geq 5\delta/6$, which is a contradiction since by construction $\sigma(Vz_n) \leq \delta/3$. Therefore $x_n - Vz_n \to 0$, and so $Vz_n \to 0$ since x_n is a null sequence.

Finally, it is clear that $\pi V = Id_H$

Step 3. To prove the theorem, choose $\gamma > 0$ such that if $|x| < \gamma$, then $|Tx| < \delta/3$. Now X has L_0-structure, so

$$X = \bigoplus_{i=1}^{n} X_i$$

where $\operatorname{diam}(X_i) < \gamma$ and hence $\sigma(x) \leq \gamma$ for each x in X_i.

Now $TX = \sum_{i=1}^{n} TX_i$ and if z is in TX_i, then $\sigma(z) \leq \delta/3$.

Let $H_i = TX_i$ for $i = 1, 2, \ldots, n$. Apply step 2 to each H_i. Then there exists a continuous operator $V_i : H_i \longrightarrow Y$ such that $\pi V_i = Id_{H_i}$. The desired lifting is

$$\tilde{T}\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} V_i(Tx_i)$$

$(x_i \in X_i$ for $i = 1, 2, \ldots, n)$

This completes the proof.

PROPOSITION: Let X be a p-convex space and N a closed subspace of X. If T is a continuous linear operator from ℓ_p^n to the quotient space X/N, then T can be lifted to an operator $\tilde{T}$ from ℓ_p^n to X such that $\|\tilde{T}\| \leq 2\|T\|$.

$$\begin{tikzcd}
\ell_p^n \arrow[r, "T"] \arrow[dr, "\tilde{T}"'] & X/N \\
& X \arrow[u, "\pi"']
\end{tikzcd}$$

Proof. Let (e_i), $1 \leq i \leq n$, denote the basis vectors of ℓ_p^n. For each $i = 1, 2, \ldots, n$ pick x_i in X such that $\|x_i\| \leq 2\|\pi(x_i)\|$ and $\pi(x_i) = Te_i$. Define $\tilde{T}(e_i) = x_i$ and extend $\tilde{T}$ to all of ℓ_p^n by linearity. Then using the p-convexity of X, we obtain

$$\begin{aligned}
\|\tilde{T}(\textstyle\sum \alpha_i e_i)\|^p &= \|\textstyle\sum \alpha_i \tilde{T} e_i\|^p \\
&\leq \textstyle\sum |\alpha_i|^p \|\tilde{T} e_i\|^p
\end{aligned}$$

$$= \sum_i |\alpha_i|^p \, \|x_i\|^p$$

$$\leq \sum_i |\alpha_i|^p \, 2^p \, \|T\|^p$$

and so

$$\|\tilde{T}(\sum \alpha_i e_i)\| \leq 2\|T\| \left(\sum |\alpha_i|^p\right)^{1/p}$$

Therefore $\tilde{T}$ is continuous and $\|\tilde{T}\| \leq 2\|T\|$.

Finally, for each $i = 1, 2, \ldots, n$, $\pi\tilde{T}(e_i) = \pi(x_i) = Te_i$, and so $\pi \circ \tilde{T} = T$. Thus $\tilde{T}$ is the desired lifting.

DEFINITION: A (p-convex) space X is a $\mathcal{L}_p$-space if

$$X = \overline{\bigcup_{\alpha \in A} X_\alpha}$$

where A is a directed set and

(i) $\alpha \leq \beta \implies X_\alpha \subset X_\beta$

(ii) each X_α is finite dimensional

(iii) if $k_\alpha = \dim X_\alpha$, then for each α in A

there exists an isomorphism S_α of X_α onto $\ell_p^{k_\alpha}$ such that $\|S_\alpha\| \|S_\alpha^{-1}\| \le C$, where C is a constant independent of α

The sequence spaces ℓ_p are all $\mathcal{L}_p$-spaces. We take A to be the set of positive integers, and for each integer n let X_n be the space of all sequences in ℓ_p supported on only the first n coordinates. The operator S_n is the restriction to the first n coordinates. It is easily seen that $\|S_n\| = 1$ and $\|S_n^{-1}\| = 1$ for each n.

The function spaces L_p are also $\mathcal{L}_p$-spaces. Again we take our index set to be the positive integers. Given an integer n, let X_n be the span of the characteristic functions of the 2^n dyadic intervals of $[0,1]$ of length $1/2^n$. Then

$$\left\| \sum_{i=1}^{2^n} \alpha_i \chi_{I_i} \right\|_p = \frac{1}{2^{n/p}} \left(\sum |\alpha_i|^p \right)^{1/p}$$

and so the operator $S_n : X_n \to \ell_p^{2^n}$ given by

$$S_n\left(\sum_{i=1}^{2^n} \alpha_i \chi_{I_i}\right) = (\alpha_1, \alpha_2, \ldots, \alpha_{2^n})$$

is an isomorphism with $\|S_n\| = 1/2^{n/p}$ and $\|S_n^{-1}\| = 2^{n/p}$. Moreover, the span of the simple functions over all dyadic intervals is dense in L_p.

The Hardy spaces H_p are not $\mathcal{L}_p$-spaces.

<u>Lifting Theorem for $\mathcal{L}_p$-spaces.</u> Let Y be a $\mathcal{L}_p$-space. Let X be p-convex and suppose N is a closed subspace of X whose unit ball is compact in some Hausdorff vector topology. Then any continuous operator $T : Y \to X/N$ has a lifting $\tilde{T} : Y \to X$.

$$\begin{tikzcd} Y \arrow[r, "T"] \arrow[dr, "\tilde{T}"'] & X/N \\ & X \arrow[u, "\pi"'] \end{tikzcd}$$

Remark: This theorem applies when N is finite dimensional or when X is the L_p space on the unit circle and N is H_p.

In the latter case, the topology of uniform convergence on compact subsets of the unit disk is compact for the unit ball of H_p.

Proof of theorem. Since Y is a $\mathcal{L}_p$-space, we can write $Y = \overline{\bigcup_{\alpha \in A} Y_\alpha}$ with the corresponding isomorphisms S_α. Let T_α denote the restriction of T to Y_α. Then $T_\alpha \circ S_\alpha^{-1}$ is a continuous operator from a finite dimensional ℓ_p space to the quotient space X/N. By the previous proposition, there is a lifting of $S^{-1} \circ T_\alpha$ to an operator R_α with

$$\|R_\alpha\| \leq 2 \|T_\alpha \circ S_\alpha^{-1}\| \leq 2\|S^{-1}\| \, \|T\|$$

Define $\tilde{T}_\alpha : Y_\alpha \to X$ by $\tilde{T}_\alpha = R_\alpha \circ S_\alpha$. Then

$$\pi \tilde{T}_\alpha = \pi R_\alpha \circ S_\alpha = T_\alpha \circ S_\alpha^{-1} \circ S_\alpha = T_\alpha$$

and so $\tilde{T}_\alpha$ is a lifting of T_α. Moreover,

$$\| \tilde{T}_\alpha \| \leq 2 \| S_\alpha^{-1} \| \, \| T \| \, \| S_\alpha \| \leq 2c \| T \|$$

Now for each y in Y let $\upsilon(y) = z$, where z satisfies

(i) $\pi(z) = Ty$

(ii) $\|z\| \leq 2\|Ty\| \leq 2\|T\| \, \|y\|$

Consider the product space

$$\prod_{\bigcup_\alpha Y_\alpha} K_y \, \mathrm{Ball}(N)$$

where $K_y = 2^{\frac{1}{p}}(c+1)\|T\| \, \|y\|$. This is compact in the product topology. For α in A define f_α in this space by

$$f_\alpha(y) = \begin{cases} \tilde{T}_\alpha(y) - \upsilon(y) & \text{if } y \in Y_\alpha \\ 0 & \text{otherwise} \end{cases}$$

Since $\pi(\tilde{T}_\alpha(y) - \upsilon(y)) = 0$, the element $\tilde{T}_\alpha(y) - \upsilon(y)$ belongs to N, and moreover

$$\|\tilde{T}_\alpha(y) - u(y)\| \leq 2^{\frac{1}{p}-1}\left(2c\|T\|\|y\| + 2\|T\|\|y\|\right)$$

$$= 2^{1/p}(c+1)\|T\|\|y\|$$

$$= K_y$$

Hence $f_\alpha(y)$ does belong to $K_y \, Ball(N)$.

Let (f_β) be a convergent subnet of (f_α), say $f_\beta \rightarrow V$.

If x, y belong to Y_β, then

$$f_\beta(x+y) - f_\beta(x) - f_\beta(y)$$

$$= \tilde{T}_\beta(x+y) - u(x+y) - \tilde{T}_\beta(x) + u(x) - \tilde{T}_\beta(y) + u(y)$$

$$= -u(x+y) + u(x) + u(y)$$

and so $V(x+y) - V(x) - V(y) = -u(x+y) + u(x) + u(y)$. If we define $\tilde{T}(y) = u(y) + V(y)$, then the above calculation shows that $\tilde{T}$ is additive. Similarly, one can show that $\tilde{T}(ky) = k\tilde{T}(y)$ for each scalar k and element y. Hence $\tilde{T}$ is linear on Y.

Moreover,

$$\|\tilde{T}(y)\| \leq 2^{\frac{1}{p}-1}\left(2\|T\|\|y\| + 2^{\frac{1}{p}-1}\left(2c\|T\|\|y\| + 2\|T\|\|y\|\right)\right)$$

$$\leq \gamma \|y\|$$

and so $\tilde{T}$ is continuous from Y to X.

To see that $\tilde{T}$ is the desired lift, observe that for each y in $\bigcup_{\alpha \in A} Y_\alpha$,

$$\pi \tilde{T}(y) = \pi U(y) + \pi V(y) = Ty + 0 = Ty$$

since $V(y)$ belongs to N.

In the case when Y is equal to $L_p[0,1]$, the lift $\tilde{T}$ is unique. To see this, suppose T_1 is another lift of T. Then the operator $U = \tilde{T} - T_1$ maps L_p into N. Consider U as a map from L_p into N with the compact τ. We claim that U is norm-to-τ continuous. If $U \neq 0$, then without loss of generality we assume $\|U\| = 1$, and so $U(\text{Ball}(L_p)) \subset \text{Ball}(N)$.

Let W be a τ-neighborhood of 0. Choose $\lambda > 0$ so that $\lambda \text{Ball}(N) \subset W$. Then $U(\lambda \text{Ball}(L_p)) \subset \lambda \text{Ball}(N) \subset W$, which establishes the claim. But now we would have a continuous compact operator $U : L_p \to (N, \tau)$ which, by Kalton's result, preserves a copy of ℓ_2. Since this is impossible for compact operators, we must have $U = 0$. Therefore the lift $\tilde{T}$ is unique.

The next theorem gives another situation where there is a unique lift.

THEOREM: Let X be a complete p-convex space and N a closed subspace of X such that the quasi-norm on N is equivalent to a q-convex quasi-norm for some $q > p$. If T is a continuous operator from L_p to X/N, then T has a unique lift $\tilde{T} : X \to X$.

Proof. We will first exhibit the uniqueness by showing that if U is a continuous operator from L_p into a q-convex space Z $(q > p)$, then $U = 0$. Suppose f belongs to L_p, and $|f| \leq M$ almost everywhere. Then for any n,

$$f = \sum_{k=1}^{n} f \chi_{\left[\frac{k-1}{n}, \frac{k}{n}\right]}$$

and so

$$Uf = \sum_{k=1}^{n} Uf \chi_{\left[\frac{k-1}{n}, \frac{k}{n}\right]} .$$

Hence

$$\|Uf\|^q \leq \sum_{k=1}^{n} \|U\|^q \|f \chi_{\left[\frac{k-1}{n}, \frac{k}{n}\right]}\|_p^q$$

$$\leq \|U\|^q M^q n^{-q/p} n$$

$$= \|U\|^q M^q n^{1-q/p}$$

Since $1 - \frac{q}{p} < 0$, and the above inequality holds for all n, we must have $\|Uf\| = 0$. It therefore follows that $U = 0$.

We now show the existence of the lift. Towards this end, write $\chi_j^n = \chi_{\Delta_j^n}$ where $\Delta_j^n = \left[\frac{j-1}{2^n}, \frac{j}{2^n}\right]$, $j=1,2,\ldots,2^n$.

Then

$$L_p = \overline{\bigcup_n Y_n}$$

where Y_n is the subspace spanned by the set $\{\chi_j^n : j=1,2,\ldots,2^n\}$

For each n there is a lift $\tilde{T}_n$ of the restriction of T to Y_n with $\|\tilde{T}_n\| \leq 2\|T\|$. Now fix j and m. We claim that the sequence $(\tilde{T}_n(\chi_j^m))_{n=m}^{\infty}$ is Cauchy in n. To see this, take $r > n \geq m$ and write

$$\chi_j^m = \sum_{i=1}^{2^{n-m}} \chi_{q_i}^n .$$

Then

$$\|\tilde{T}_r(\chi_j^m) - \tilde{T}_n(\chi_j^m)\|$$

$$= \left\| \sum_{i=1}^{2^{n-m}} (\tilde{T}_r - \tilde{T}_n)\, \chi_{q_i}^n \right\|$$

$$\leq K\Big(\sum_{i=1}^{2^{n-m}} \|(\tilde{T}_r - \tilde{T}_n)\chi_{q_i}^n\|^q \Big)^{1/q}$$

$$\leq K \; \|\tilde{T}_r - \tilde{T}_n\| \Big(\sum_{i=1}^{2^{n-m}} \|\chi_{q_i}^n\|_p^q \Big)^{1/q}$$

$$= K \|\tilde{T}_r - \tilde{T}_n\| \big((2^n)^{-q/p} \, 2^{n-m} \big)^{1/q}$$

$$= K \|\tilde{T}_r - \tilde{T}_n\| \big(2^{n(1-q/p)} \, 2^{-m} \big)^{1/q}$$

$$\leq 4K \, 2^{\frac{1}{p}-1} \|T\| \big(2^{n(1-q/p)} \, 2^{-m} \big)^{1/q}$$

and this last term converges to 0 as $n \to \infty$, since $q > p$. Since X is complete, the sequence $(\tilde{T}_n(\chi_j^m))$ converges to some element of X. Hence $(\tilde{T}_n(y))$ converges for each y in $\bigcup_{n=1}^{\infty} Y_n$. Let $\tilde{T}(y) = \lim \tilde{T}_n(y)$. Then $\tilde{T}$ is linear and continuous, and so has a continuous linear extension $\tilde{T}$ to all of L_p, which is the desired lift.

Applications of Liftings

(1) There exists a continuous linear operator T from H_p onto ℓ_p $(0<p\leq 1)$ of the form

$$Tf = \left((1-|z_n|^2)^{1/p} f(z_n) \right)$$

where (z_n) in the open unit disk is a "uniformly separated" sequence (see Duren). Let N be the kernal of T. Under the topology of uniform convergence on compact sets, the ball of N is compact. Therefore the isomorphism S from ℓ_p to H_p/N has a lift $\tilde{S}$ from ℓ_p to H_p, which shows that ℓ_p is a complemented subspace of H_p.

(2) Suppose X is a Hausdorff topological vector space and R is a one-dimensional subspace of X such that the quotient space X/R is isomorphic to ℓ_p. Let $T: \ell_p \to X/R$ be the isomorphism. If X were locally bounded and

p-convex, we could then apply either lifting theorem to T to obtain an isomorphism from ℓ_p into X. (It will be shown later that this is indeed the case since (i) if X/B and B are both locally bounded, then X is locally bounded, and (ii) if X/B is p-convex and B is q-convex for $q > p$, then X is p-convex)