Vector Measures

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INTEGRATION & RNP

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1/30 VECTOR MEASURES

Let (Ω, Σ, μ) be a finite measure space
X Banach space

DEFINITION: A function of the form $f = \sum_{i=1}^{n} x_i \chi_{E_i}$ where $x_i \in X$ and $E_i \in \Sigma$ is called a simple function

$$f : \Omega \longrightarrow X$$

DEFINITION: A function $f : \Omega \to X$ is called μ-measurable (measurable - strongly measurable - norm measurable) if there exist a sequence (f_n) of simple functions such that

$$\|f_n - f\|_X \longrightarrow 0 \quad a.e.$$

The function f is called weakly (scaler) measurable if $x^* f$ is measurable $\forall x^* \in X^*$

If $\Gamma \subset X^*$ and if $x^* f$ is measurable $\forall x^* \in \Gamma$, we say f is Γ measurable

THEOREM (PETTIS) A function $f : \Omega \to X$ is measurable iff it is weakly measurable and almost separably valued

$\uparrow$ $f(\Omega \setminus E)$ is separable where $\mu(\Omega \setminus E) = 0$

Proof. Assume f is measurable. Let (f_n) be simple s.t

$f_n \to f$ a.e. in norm. Let E be the exceptional set, so $\mu(E)=0$ and $f_n(\omega) \to f(\omega)$ $\forall \omega \notin E$. Accordingly

$$f(\Omega\backslash E) \subset \underbrace{\overline{\bigcup_{n=1}^{\infty} f_n(\Omega\backslash E)}}_{\substack{\text{this is a countable set} \\ \text{so closure is separable}}}$$

Hence f is essentially separably valued.

To see that f is weakly measurable, let $x^* \in X^*$ and notice $x^* f_n \to x^* f$ pointwise off E, and so $x^* f$ is measurable.

Conversely, let E be a null set such that $f(\Omega\backslash E)$ is separable. Take $(x_n) \subset \overline{sp}\, f(\Omega\backslash E)$ which is dense in $\overline{sp}\, f(\Omega\backslash E)$. By Hahn-Banach there is a sequence $(x_n^*) \subset X^*$ with $\|x_n^*\| = 1$, $x_n^*(x_n) = \|x_n\|$. Notice that

$$\|f(\omega)\| = \sup_n |x_n^* f(\omega)| \quad \forall \omega \notin E$$

Better yet, notice that $\forall \omega \in \Omega\backslash E$

$$\|f(\omega) - x_m\| = \sup_n \overbrace{|x_n^*(f(\omega) - x_m)|}^{\text{measurable by hypothesis}}$$

Put $g_m(\omega) := \|f(\omega) - x_m\|$ $\forall \omega$. As the supremum of a sequence of measurable scaler-valued functions, the real valued non-negative function g_m is measurable.

Let $\varepsilon > 0$. Set

$$E_m := \{\omega : g_m(\omega) < \varepsilon\}^{\setminus E} = [g_m < \varepsilon] \setminus E$$

Since (x_n) is dense in $f(\Omega \setminus E)$, we see that

$$\bigcup_{m=1}^{\infty} E_m = \Omega \setminus E$$

Define for $\omega \in E$, $f_\varepsilon = 0$; otherwise set

$$f_\varepsilon(\omega) := x_n \text{ if } \omega \in E_n \setminus \bigcup_{k=1}^{n-1} E_k$$

and observe that $\| f(\omega) - f_\varepsilon(\omega) \| < \varepsilon \quad \forall \omega \in \Omega \setminus E$. This proves that f is the a.e. uniform limit of countably valued functions.

To finish the proof, we must find a seq of simple functions φ_n s.t. $\varphi_n \longrightarrow f$ a.e. To do this, for each n put

$$f_{1/n} = \sum_{m=1}^{\infty} x_{n,m} \chi_{\underset{\text{disjoint sets}}{E_{n,m}}}$$

For each n select p_n s.t.

$$\mu\left(\bigcup_{m=p_n+1}^{\infty} E_{n,m} \right) < 1/2^n$$

Put

$$\varphi_n := \sum_{m=1}^{p_n} x_{n,m} \chi_{E_{n,m}} \quad \text{(simple functions)}$$

Then $\|\varphi_n - f\|$ converges almost uniformly to zero. Hence f is measurable.
☆(See next page)

$\blacksquare$

Corollary (of proof) A function $f : \Omega \to X$ is measurable iff it is the a.e. unif. limit of a sequence of countably valued (measurable) functions

Corollary (of proof) Let f be an essentially separably valued function and suppose $\exists$ a countable norming set $\Gamma \subset X^*$ such that $x^* f$ is measurable $\forall x^* \in \Gamma$. Then f is measurable.

(take x_n^* in proof from Γ)

prel.!! HW – ① Let $f : \Omega \to X$ be weakly measurable and suppose $\exists$ seq. of simple functions f_n and a null set E s.t.

$$\omega \notin E \implies f_n(\omega) \to f(\omega) \text{ weakly}$$

weak separability = norm separability by H–B

Then f is measurable (Show $f(\Omega \setminus E)$ is separably valued)

② Let μ be Lebesgue measurable on $[0,1]$. TFAE

(i) $f : [0,1] \to X$ is measurable

(ii) $\forall \varepsilon > 0$ $\exists$ compact $K \subset [0,1]$ s.t. $f|_K$ is continuous (with $\mu([0,1] \setminus K) < \varepsilon$) (Lusin measurability)

(iii) $\forall \varepsilon > 0$ $\exists$ a compact K with $\mu([0,1] \setminus K) < \varepsilon$ s.t. $f|_K$ is weakly continuous (f sep. valued) (weak Lusin measurability)

$s_n \to s$ in $K \Rightarrow x^* f(s_n) \to x^* f(s)$ $\forall x^*$

★ Let $\varepsilon > 0$ and choose n_0 $\sum_{k=n_0+1}^{\infty} 1/2^k < \varepsilon$. If $\omega \notin \bigcup_{m=n_0+1}^{\infty} \bigcup_{k=p_m+1}^{\infty} E_{m,k}$

then if $\delta > 0$ and $n > n_1 > \max(1/\delta, n_0)$

$$\|f(\omega) - \varphi_n(\omega)\| \le \|f(\omega) - f_{1/n}(\omega)\| + \|f_{1/n}(\omega) - \varphi_n(\omega)\|$$

$$\le 1/n < \delta$$

$\Rightarrow \|f - \varphi_n\| \to 0$ unif on $\Omega \setminus \bigcup_{m=p_{n_0}+1}^{\infty} E_{n_0,m}$.

Lemma: If $f_n \to f$ a.u., then $f_n \to f$ a.e.

Proof. For each m choose A_m s.t. $\mu(A_m) < 1/m$ and $f_n \to f$ uniformly off A_m. Let $B = \bigcap_{m=1}^{\infty} A_m$. Then $\mu(B) < 1/m$ $\forall m \Rightarrow \mu(B) = 0$. Let $\varepsilon > 0$ and $x \notin B$. Then $\exists m_0$ s.t. $x \notin A_{m_0}$. Since $f_n \to f$ off A_{m_0}, $\exists n_0$ s.t. $|f(x) - f_n(x)| < \varepsilon$ $\forall n > n_0$. Therefore $f_n \to f$ off B.

Examples

① (Birkhoff) A weakly measurable function that is not measurable

$$X = \ell_2[0,1]$$

Let $\{e_t\}_{t\in[0,1]}$ be a complete orthonormal system

(For each $t \in [0,1]$ define a vector e_t. Define norm

$$\left\| \sum_{n=1}^{m} \alpha_n e_{t_n} \right\| = \sqrt{\sum \alpha_n^2}$$

$\ell_2[0,1]$ = completion of this space. Alternatively consider all functions $f:[0,1] \to \mathbb{R}$ with the property

$$\lim_{\Delta} \sqrt{\sum_{t\in\Delta} f(t)^2} < \infty$$

where the sum is taken over all finite subsets Δ of $[0,1]$ directed by set inclusion)

Define $f:[0,1] \to \ell_2[0,1]$ by $f(t) = e_t$. Then f is weakly measurable since if $x^* \in X^* = \ell_2[0,1]^* = \ell_2[0,1]$ then

$$x^* = \sum_{n=1}^{\infty} \beta_n e_{t_n}$$

and $x^* f(t) = \beta_n$ if $t = t_n$
$\quad = 0$ otherwise

Hence x^*f is 0 except on a countable set

Hence $x^* f$ is Lebesgue measurable. However, if $t \neq t'$, then

$$\| f(t) - f(t') \| = \| e_t - e_{t'} \| = \sqrt{2}$$

Hence $f(A)$ is separable iff A is countable. Hence f is not essentially separably valued.

(2) A weakly measurable function that is not measurable.

Define $f : [0,1] \to L_\infty[0,1]$ by $f(t) = \chi_{[0,t]}$. To understand why f is weakly measurable, one must be "comfortable" in $L_\infty(\mu)^*$

[For a general finite measure μ, $L_\infty(\mu)^*$ = space of all bounded finitely additive measures λ on Σ that vanish on μ-null sets

$$\langle \lambda, f \rangle = \int f \, d\lambda$$

The important thing to know for this example is that each $\lambda \in L_\infty(\mu)^*$ can be written as $\lambda = \lambda^+ - \lambda^-$ where λ^+, λ^- are non-negative in $L_\infty(\mu)^*$ (Jordan Decomposition Theorem)]

Take $\lambda \in L_\infty[0,1]^*$ and write $\lambda = \lambda^+ - \lambda^-$. Then fix $t \in [0,1]$.

$$\langle \lambda, f(t) \rangle = \int_\Omega f(t) \, d\lambda = \int_\Omega f(t) \, d\lambda^+ - \int_\Omega f(t) \, d\lambda^-$$

$$= \int \chi_{[0,t]} \, d\lambda^+ - \int \chi_{[0,t]} \, d\lambda^- = \lambda^+[0,t] - \lambda^-[0,t]$$

monotone increasing functions of t

Hence this is a measurable function of t

Hence $\langle \lambda, f(t) \rangle$ is a measurable function of t $\forall \lambda \in L_\infty[0,1]^*$
$\Rightarrow$ f is weakly measurable.

This function is not measurable because if $t \neq t'$ then $\|f(t) - f(t')\|_\infty = 1$. Hence $f(A)$ is separable iff A is countable.
Hence f is not essentially separable valued

2/1 VECTOR MEASURES

Example (Sierpinski) A function into a dual which is w^*-measurable but not weakly measurable.

i.e. $f : \Omega \to X^*$ $\quad x \cdot f$ is measurable $\forall x \in X$ but $\exists x^{**} \in X^{**}$ s.t. $x^{**} f$ is not measurable. Notice X must be s.t. X is not w^*-sequentially dense in X^{**}. For if X is w^*-seq. dense $\forall x^{**} \in X^{**}$ $\exists$ a sequence (x_n) in X s.t.

Can always do this with nets by Goldstine

$$x^{**}(x^*) = \lim_n x^*(x_n) \quad \forall x^* \in X^*$$

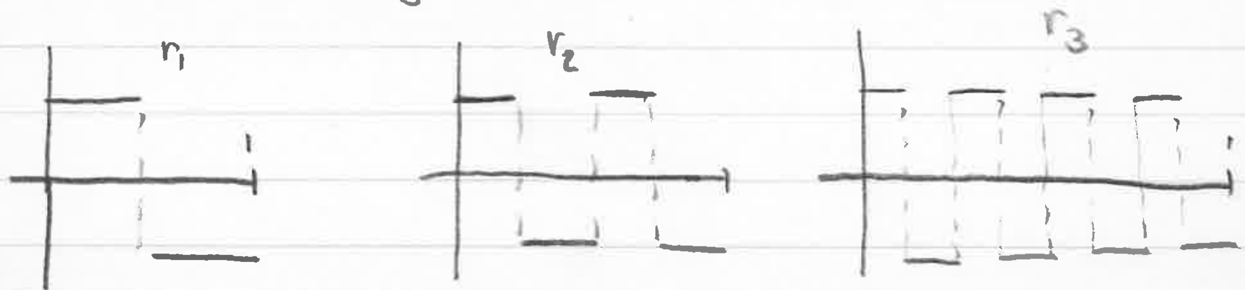
$$\Rightarrow x^{**} f = \lim x_n f \text{ pointwise}$$
$$\Rightarrow x^{**} f \text{ is measurable}$$

Hence f w^*-measurable into X^*, X w^*-seq. dense in X^{**} $\Rightarrow f$ weakly measurable

If X is separable, this is equivalent to ℓ_1 is not isomorphic to a subspace of X

Back to example - Let r_n = n^{th} Rademacher function

$$r_n(t) = \operatorname{sgn}(\sin(2^n \pi t)) \quad 0 \le t \le 1$$



Define $f : [0,1] \longrightarrow \ell_\infty$ by

$$f(t) = \left(\frac{r_n(t)+1}{2} : n \in \mathbb{N} \right)$$

Note that $\frac{r_n(t)+1}{2}$ is $\{0,1\}$ valued. Notice

$$\| f(t) - f(t') \|_\infty = 1 \quad \forall \text{ non-dyadic } t', t \text{ with } t \neq t'$$

$$\Longrightarrow f \text{ is not essentially separably valued}$$

Is f w^*-measurable? Take $x \in \ell_1$ ($\ell_\infty = \ell_1^*$) $x = (\alpha_n)$ with $\sum |\alpha_n| < \infty$. Then

$$x \cdot f(t) = \sum_{n=1}^{\infty} \alpha_n \left(\frac{r_n(t)+1}{2} \right) \quad \text{measurable since convergent}$$

To see f is not weakly measurable we'll look at ℓ_∞^* = all bounded finitely additive measures on $\mathcal{P}(\mathbb{N})$. Let $\beta \in \ell_\infty^*$ be a $\{0,1\}$ purely finitely additive measure, i.e.

$$\beta(\mathbb{N}) = 1$$
$$\beta(\text{finite set}) = 0$$
$$\beta(\text{any set}) = 0 \text{ or } 1$$

(~~for example Banach limits~~)

Notice

$$f(t) + f(1-t) = (1,1,1,1,\dots)$$

since $r_n(t) + r_n(1-t) = 0 \ \forall n$ (non-dyadic). Let β correspond to $x^{**} \in \ell_\infty^*$. Then

$$x^{**} f(t) = \int_{\mathbb{N}} f(t)\, d\beta = \varphi(t)$$

Claim: $\varphi(t)$ is not measurable. Notice

$$\varphi(t) = \begin{cases} 0 & \text{if } \beta\left(\left\{n : \frac{r_n(t)+1}{2} = 1\right\}\right) = 0 \\ 1 & \text{if } \beta\left(\left\{n : \frac{r_n(t)+1}{2} = 1\right\}\right) = 1 \end{cases}$$

mod 1

Also, since β vanishes on finite subsets of $\mathbb{N}$, $\varphi(t) = \varphi(t+d)$ for all dyadic d and non-dyadic t. $f(t)$ and $f(t+d)$ differ only a finite number of coordinates. Hence φ has a dense set of periods. Hence if φ is measurable, it is a.e. constant, so $\varphi = k$ a.e. and $k = 0$ or 1. But

$$\varphi(1-t) = \int_{\mathbb{N}} f(1-t)\, d\beta = \int_{\mathbb{N}} [(1,1,1,\dots) - f(t)]\, d\beta$$

$$= \int_{\mathbb{N}} (1,1,1,\dots)\, d\beta - \int f(t)\, d\beta = \underset{\beta(\mathbb{N})}{1} - \varphi(t)$$

$$\Rightarrow K = 1 - K \Rightarrow 2K = 1 \Rightarrow K = 1/2 \ \lightning$$

★ $f(t) = \chi_E$ where $E = \{n : \frac{r_n(t)+1}{2} = 1\}$. Then

$$\int_{\mathbb{N}} f(t)\,\partial\beta = \beta(E) = 0 \text{ or } 1$$

Let $E_1 = \{n : f(t)(n) \neq f(t+d)(n)\}$. E_1 is finite. Then

$$\varphi(t) = \underbrace{\int_{E_1} f(t)\,\partial\beta}_{0 \text{ since } \beta(E_1)=0} + \int_{\mathbb{N}\setminus E_1} f(t)\,\partial\beta = \int_{\mathbb{N}\setminus E_1} f(t+d)\,\partial\beta = \int_{\mathbb{N}} f(t+d)\,\partial\beta = \varphi(t+d)$$

INTEGRATION

<u>Bochner integral</u> (Dunford-Schwartz integral, Dunford's 1st integral)

Let (Ω, Σ, μ) be a finite measure space.

<u>DEFINITION</u>: A measurable function $f : \Omega \to X$ is called Bochner integrable if $\exists$ a seq. (f_n) of simple (X-valued) functions such that $\|f - f_n\|_X \in L_1(\mu)$ $\forall n$ and

$$\lim_n \int_\Omega \|f - f_n\| \, d\mu = 0$$

In this case we define

$$\int_E f \, d\mu := \lim_n \int_E f_n \, d\mu$$

(norm limit in X)

To see that this limit exists, look at

$$\left\| \int_E f_n \, d\mu - \int_E f_m \, d\mu \right\| = \left\| \int_E (f_m - f_n) \, d\mu \right\|$$

$$\leq \int_E \|f_n - f_m\| \, d\mu \leq \int_\Omega \|f_n - f\| d\mu + \int_\Omega \|f - f_m\| d\mu \to 0$$

Hence $\left(\int_E f_n \, d\mu \right)$ is a Cauchy sequence in X.

HW Let f be Bochner integrable. Show that $\{ \int_E f \, d\mu : E \in \Sigma \}$ is relatively norm compact (Show total bounded)

<u>**THEOREM**</u>: A measurable function $f : \Omega \to X$ is Bochner integrable if and only if $\|f\|_X \in L_1(\mu)$

Proof $(\Rightarrow)$

$$\int_\Omega \|f\| \, d\mu \leq \underbrace{\int_\Omega \|f - f_n\| \, d\mu}_{\to 0} + \underbrace{\int_\Omega \|f_n\| \, d\mu}_{< \infty} < \infty$$

$(\Leftarrow)$ Let f be measurable with $\|f\| \in L_1(\mu)$. Choose a sequence of countably valued functions (f_n) s.t.

$$\| f - f_n \| \leq 1/n \quad \text{a.e.}$$

Notice that $\|f_n\| \leq \|f\| + 1/n$, so $\|f_n\| \in L_1(\mu)$.
Write

$$f_n = \sum_{m=1}^{\infty} x_{m,n} \chi_{E_{m,n}}$$

disjoint

Pick p_n so large that

$$(*) \qquad \int_{\bigcup_{p_n+1}^{\infty} E_{m,n}} \|f_n\| \, d\mu < \frac{\mu(\Omega)}{n}$$

Let

$$g_n := \sum_{m=1}^{p_n} x_{m,n} \chi_{E_{m,n}}$$

Then each g_n is a simple function and

$$\int_{\Omega} \|f - g_n\| \, d\mu \leq \int_{\Omega} \|f - f_n\| \, d\mu + \int_{\Omega} \|f_n - g_n\| \, d\mu$$

$$\leq \frac{\mu(\Omega)}{n} + \int_{\bigcup_{p_n+1}^{\infty} E_{m,n}} \|f_n\| \, d\mu \leq \frac{2\mu(\Omega)}{n} \to 0$$

Hence f is Bochner integrable.

$\blacksquare$

<u>Easy properties of Bochner integral</u> : Let f be Bochner integrable

☆ Ⓐ $\left\| \int_E f \, d\mu \right\| \leq \int_E \|f\| \, d\mu$

☆ Do Ⓐ for simple functions : $\left\| \int_E f \, d\mu \right\| = \left\| \int_E \sum \alpha_n \chi_{E_n} \, d\mu \right\|$

$= \left\| \sum \alpha_n \mu(E \cap E_n) \right\| \le \sum \|\alpha_n\| \mu(E \cap E_n) = \int_E \sum \|\alpha_n\| \chi_{E_n} \, d\mu$

$= \int_E \|f\| \, d\mu$

(B) $\lim\limits_{\mu(E)\to 0} \int_E f\,d\mu = 0$

Proof. $\lim\limits_{\mu(E)\to 0} \left\| \int_E f\,d\mu \right\| \leq \lim\limits_{\mu(E)\to 0} \int_E \|f\|\,d\mu = 0$ ← 441 thm

(C) If (E_n) is a disjoint seq. in Σ, then

$$\int_{\bigcup_{n=1}^{\infty} E_n} f\,d\mu = \sum_{n=1}^{\infty} \int_{E_n} f\,d\mu$$

where the series on the right is absolutely convergent.

Proof. To see the absolute convergence, notice

$$\sum_{n=1}^{\infty} \left\| \int_{E_n} f\,d\mu \right\| \leq \sum_{n=1}^{\infty} \int_{E_n} \|f\|\,d\mu = \int_{\cup E_n} \|f\|\,d\mu < \infty$$

To see that it converges to the right thing, notice that

$$\int_{\bigcup_{n=1}^{k} E_n} f\,d\mu = \sum_{n=1}^{k} \int_{E_n} f\,d\mu$$

Hence

$$\int_{\bigcup_{n=1}^{\infty} E_n} f\,d\mu = \int_{\bigcup_{n=1}^{k} E_n} f\,d\mu + \int_{\bigcup_{n=k+1}^{\infty} E_n} f\,d\mu$$

$$= \sum_{n=1}^{k} \int_{E_n} f\,d\mu + \underbrace{\int_{\bigcup_{n=k+1}^{\infty} E_n} f\,d\mu}_{\text{tends to } 0 \text{ as } k\to\infty \text{ since } \mu\left(\bigcup_{n=k+1}^{\infty} E_n\right) \to 0}$$

$\blacksquare$

DEFINITION: For a finitely additive function $F : \Sigma \to X$ and $E \in \Sigma$, we define the variation $|F|(E)$ of F on E by

$$|F|(E) = \sup_{\pi} \sum_{A\in\pi} \|F(A)\|$$

where the sup is taken over all partitions π of E into a finite collection of disjoint sets in Σ whose union is E.

FACT - If π and π' are partitions of E s.t. each $A \in \pi \Rightarrow A$ is a union of members of π' (i.e. $\pi \le \pi'$ or π' refines π) then

$$\sum_{A\in\pi} \|F(A)\| \le \sum_{B\in\pi'} \|F(B)\|$$

$$(\|F(B_1 \cup B_2)\| = \|F(B_1) + F(B_2)\| \leq \|F(B_1)\| + \|F(B_2)\|)$$

Hence

$$|F|(E) = \lim_{\pi} \sum_{A \in \pi} \|F(A)\|$$

(D) Let $F(E) := \int_E f \, d\mu$ for $E \in \Sigma$. Then $|F|(E) = \int_E \|f\| \, d\mu$

Proof. First prove $|F|(\Omega) < \infty$. Let π be a partition of Ω

$$\sum_{A \in \pi} \|F(A)\| = \sum_{A \in \pi} \left\| \int_A f \, d\mu \right\| \leq \sum_{A \in \pi} \int_A \|f\| \, d\mu$$

$$= \int_\Omega \|f\| \, d\mu < \infty$$

Now let (f_n) be a "defining" sequence for f. Let $\varepsilon > 0$ and fix n_0 s.t.

$$n \geq n_0 \implies \int_\Omega \|f - f_n\| \, d\mu < \varepsilon$$

Choose a partition π of E s.t.

$$\sum_{A \in \pi} \left\| \int_A f_{n_0} \, d\mu \right\| = \int_E \|f_{n_0}\| \, d\mu$$

Choose a partition π' of E s.t. $\pi' \geq \pi$ and

$$\left| |F|(E) - \sum_{B\in\pi'} \left\| \int_B f\,d\mu \right\| \right| < \varepsilon$$

Also $\int_E \|f_{n_0}\|\,d\mu = \sum_{B\in\pi'} \left\| \int_B f_{n_0}\,d\mu \right\|$ and

$$\sum_{B\in\pi'} \left| \left\| \int_B f\,d\mu \right\| - \left\| \int_B f_{n_0}\,d\mu \right\| \right|$$

$$\leq \sum_{B\in\pi'} \int_B \|f - f_{n_0}\|\,d\mu = \int_E \|f - f_{n_0}\|\,d\mu < \varepsilon$$

Hence

$$\left| |F|(E) - \int_E \|f_{n_0}\|\,d\mu \right| = \left| |F|(E) - \sum_{B\in\pi'} \left\| \int f_{n_0} \right\| d\mu \right| < 2\varepsilon$$

Since this is true for all large n_0, we see

$$|F|(E) = \lim_{n_0} \int_\Omega \|f_{n_0}\|\,d\mu = \int_\Omega \|f\|\,d\mu$$

2/6 VECTOR MEASURES

COROLLARY: If f, g Bochner integrable with $\int_E f\,d\mu = \int_E g\,d\mu \ \forall E \in \Sigma$, then $f = g$ a.e.

Proof. Define

$$F(E) := \int_E (f-g)\,d\mu$$

so $F = 0$. Hence

$$0 = |F|(\Omega) = \int \|f-g\|\,d\mu$$

$$\Longrightarrow \|f-g\| = 0 \quad \text{a.e.}$$

$\blacksquare$

FACT: Let F be Bochner integrable and let $T : X \to Y$ be a bounded linear operator. Then Tf is Bochner integrable and

$$\int_E (Tf)\,d\mu = T\left(\int_E f\,d\mu\right)$$

Proof. Take a sequence (f_n) of simple functions s.t $\int \|f_n - f\|\,d\mu \to 0$. Obviously

$$T\left(\int_E f_n\,d\mu\right) = \int_E Tf_n\,d\mu$$

and $T(f_n)$ is also a simple function. Moreover,

$$\int \|T(f_n) - T(f)\| \, d\mu \leq \int \|T\| \, \|f_n - f\| \, d\mu \longrightarrow 0$$

so that Tf is Bochner integrable. Finally

$$\int_E Tf \, d\mu = \lim_n \int_E T f_n \, d\mu = \lim_n T\left(\int_E f_n \, d\mu\right)$$

$$= T\left(\lim \int f_n \, d\mu\right) = T\left(\int_E f \, d\mu\right)$$

$\blacksquare$

<u>**Corollary**</u>: f, g measurable, $x^* f = x^* g$ a.e. $\forall x^*$ (a.e. can vary with x^*) implies $f = g$ a.e.

Proof. Select an expanding sequence $(E_n) \uparrow \Omega$ in Σ s.t. both $\|f\|$ and $\|g\|$ are bounded on E_n. Then both $f\chi_{E_n}$ and $g\chi_{E_n}$ are Bochner integrable (since $\|f\chi_{E_n}\| \in L_1(\mu)$ by boundedness). If $x^* \in X^*$ and $E \in \Sigma$,

$$x^* \int_E f\chi_{E_n} \, d\mu = \int_E x^* f \chi_{E_n} = \int_E x^* g \chi_{E_n} \, d\mu = x^* \int_E g \chi_{E_n} \, d\mu$$

Hence

$$\int_E f\chi_{E_n} \, d\mu = \int_E g\chi_{E_n} \, d\mu \quad \forall E \in \Sigma$$

and so $f\chi_{E_n} = g\chi_{E_n}$ a.e. $\forall n$. Since $E_n \uparrow \Omega$, this means $f = g$ a.e.

$\blacksquare$

Example: If $f : [0,1] \longrightarrow \ell_2[0,1]$ is defined by $f(t) = 0$ and $g : [0,1] \longrightarrow \ell_2[0,1]$ is defined by $g(t) = e_t$, then $x^* g = 0$ a.e.

$$\Longrightarrow \quad x^* f = x^* g \quad \text{a.e.}$$

But $f \neq g$ anywhere. This can not generalize to weak measurability

<u>COROLLARY</u>: f Bochner integrable, $E \in \Sigma$, $\mu(E) > 0 \Longrightarrow$

$$\frac{\int_E f \, d\mu}{\mu(E)} \in \overline{co}\,(f(E))$$

Proof. Suppose not. Then $\exists\, E \in \Sigma$ s.t.

$$\frac{\int_E f \, d\mu}{\mu(E)} \notin \overline{co}\,(f(E))$$

Choose with help of H-B an $x^* \in X^*$ such that

$$\frac{1}{\mu(E)}\, x^* \int_E f \, d\mu \;\leq\; \alpha \;<\; \inf x^*\left(\overline{co}\,(f(E))\right)$$

But

$$\frac{\int_E x^* f \, d\mu}{\mu(E)} > \frac{\int_E \alpha \, d\mu}{\mu(E)} = \alpha \geq \frac{x^* \int_E f \, d\mu}{\mu(E)} \quad \lightning$$

since on E $x^* f(x) > \alpha$

$\blacksquare$

<u>THEOREM</u> (KREIN-SMULIAN THEOREM): The closed convex hull of a weakly compact set in a Banach space is weakly compact

Proof Eberlein's theorem ☆ says it is enough to prove that the convex hull of W is weakly sequentially compact. A moment's reflection shows WLOG that W is separable

Consider $f: W \longrightarrow X$ defined by $f(x) = x$. This function is separably valued and weakly continuous. <u>Therefore f is μ-measurable for all regular Borel measures μ on W (weak topology)</u> f is also bounded since W is bounded. Hence f is μ-Bochner integrable $\forall \mu \in C(W)^*$ = regular Borel measures on W

Define $T: C(W)^* \longrightarrow X$ by

$$T(\mu) = \int_W f \, d\mu$$

Let $\mu_\alpha \longrightarrow \mu$ weak*, i.e.

every seq. in A has a subseq. which converges weakly to an element of X

☆ Eberlein's Theorem: A weakly sequentially compact iff the weak closure of A is weakly compact

Let $(p_n) \subset co(W)$. Each p_n is a convex combination of a finite set B_n of points of W. Let $B = \bigcup B_n$ and $X_0 = \overline{sp}(B_0)$ X_0 is separable
Let $W_0 = W \cap X_0$. Then W_0 is weakly compact and $(p_n) \subset co(W_0)$
Suffices to show $co(W_0)$ is weakly compact, where W_0 is separable

$$\int_W \varphi \, d\mu_\alpha \longrightarrow \int_W \varphi \, d\mu \qquad \forall \varphi \in C(W)$$

Let $x^* \in X^*$. Notice

$$x^*(T\mu_\alpha) = x^* \int_W f \, d\mu_\alpha = \int_W \underbrace{x^* f}_{\text{continuous}} \, d\mu_\alpha \longrightarrow \int_W x^* f \, d\mu$$

$$= x^* \int_W f \, d\mu = x^*(T\mu)$$

Hence $T : C(W)^* \longrightarrow X$ is weak* - weakly continuous

Let S = closed unit ball of $C(W)^*$. Then S is w^* compact, and so $T(S)$ is weakly compact and convex.

Fix $x \in W$. Let ε_x be the atomic pointmass at x

$$\varepsilon_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

and notice that

$$T(\varepsilon_x) = \int_W f \, d\varepsilon_x = f(x) = x$$

Hence $W \subseteq T(S) \Rightarrow \overline{co}(W) \subset T(S) \Rightarrow \overline{co}(W)$ is weakly compact.

$\blacksquare$

☆ If $f = \sum x \chi_{E_x}$, then

$$\int f \, d\varepsilon_y = \sum x \, \varepsilon_y(E_x) = \begin{cases} x & y \in E_x \\ 0 & y \notin \cup E_x \end{cases} = f(y)$$

If f Bochner integrable $\exists (f_n)$ simple s.t. $\int_\Omega \|f - f_n\| \, d\varepsilon_y \to 0$

Define

$$g_n(x) = \begin{cases} f_n(x) & x \neq y \\ f(y) & x = y \end{cases}$$

Then g_n is simple

$$\int_\Omega \| f - g_n \| \, d\varepsilon_y = \int_{\Omega \setminus \{y\}} \|f - f_n\| \, d\varepsilon_y \leq \int_\Omega \|f - f_n\| \, d\varepsilon_y \to 0$$

Then

$$\int_\Omega f \, d\varepsilon_y = \lim \int g_n \, d\varepsilon_y = \lim g_n(y) = f(y)$$

[Easier! $\displaystyle\int_\Omega f \, d\varepsilon_y = \underbrace{\int_{\Omega \setminus \{y\}} f \, d\varepsilon_y}_{0} + \int_{\{y\}} f \, d\varepsilon_y = f(y) \, \varepsilon_y\{y\} = f(y)$]

since $\varepsilon_y(\Omega \setminus \{y\}) = 0$

<u>THEOREM</u> (ORLICZ - PETTIS THEOREM) If $\sum x_n$ is a series in a Banach space with the property that each of its subseries is weakly convergent, then $\sum x_n$ and each of its subseries is norm convergent

Proof (Uhl) Suppose $\sum x_n$ is not convergent. Pick

$$n_1 < n_2 < \cdots < n_j < \cdots$$

and $\delta > 0$ s.t.

$$\left\| \sum_{n=n_j}^{n_{j+1}-1} x_n \right\| > \delta \qquad \text{(partial sums not Cauchy)}$$

Set

$$y_j = \sum_{n=n_j}^{n_{j+1}-1} x_n$$

Then $\sum y_j$ is weakly unconditional convergent (i.e. each of its subseries is weakly convergent)

Put $G = \{-1,1\}^{\mathbb{N}}$ with product topology. Let λ be the Haar measure on G

$$\text{main property: } \lambda\Big(\underbrace{\prod_{i=1}^{n-1}\{-1,1\} \times \{1\} \times \prod_{i=n+1}^{\infty}\{-1,1\}}_{E_n}\Big) = 1/2$$

☆ Let G be a topological group. A regular measure μ on G is a Haar measure on G if $\mu(sE) = \mu(E)$ $\forall s \in G$ $\forall E \in \Sigma$ (left invariance)

Haar's Theorem: Let G be a compact topological group, and let Σ be the Borel sets of G. Then there exists a regular measure on Σ which is left-invariant and not identically zero. Any two left invariant measures differ only be a scaler factor.

Define for $\vec{\varepsilon} = (\varepsilon_n) \in G$

$$f(\vec{\varepsilon}) = w\text{-}\sum_{n=1}^{\infty} \varepsilon_n y_n$$

A moment's reflection shows that $f : G \longrightarrow X$ (weak top) is continuous. f is also separably valued since $f(G) \subseteq \overline{sp}\{x_1, x_2, \dots\}$. Moreover $f(G)$ is weakly compact $\Rightarrow$ bounded. Hence f is λ-Bochner integrable

$$\int_{E_n} f \, d\lambda = \frac{y_n}{2}$$

By HW problem, $\{y_n\}_{n=1}^{\infty}$ is relatively compact. Since $\sum y_n$ is weakly convergent, $y_n \to 0$ weakly. Hence $y_n \to 0$ in norm ↯ since $\|y_n\| \geq \delta$

$\blacksquare$

The Radon-Nikodym Theorem fails for Bochner integral

Example - A c_0-valued measure F whose variation is bounded and continuous w.r.t. Lebesgue measure without a Bochner integrable derivative

Let r_n = n^{th} Radamacher function. Define

$$F(\underset{\text{Borel in }[0,1]}{E}) = \left(\int_E r_n \, d\mu \right)$$

Since $\sum y_n$ is weakly unconditionally convergent, $\sum |x^* y_n| < \infty \; \forall x^* \in X^*$ (consider the two sets where it is positive and negative) Let $W(F, \varepsilon)$ be a weak neighborhood of 0. Choose $m \in \mathbb{N}$ s.t.

finite set!

$$\sum_{n=m+1}^{\infty} |x^* y_n| < \varepsilon/2 \quad \forall x^* \in F.$$

Let $U = \{\varepsilon_1\} \times \cdots \times \{\varepsilon_m\} \times \{-1, 1\} \times \cdots$. Then U is a neighborhood of $\vec{\varepsilon} = (\varepsilon_n)$. If $\vec{\delta} = (\delta_n) \in U$, then

$$|x^*(f(\vec{\varepsilon}) - f(\vec{\delta}))| = \left| x^*\left(\sum_{n=1}^{\infty} \varepsilon_n y_n - \sum_{n=1}^{\infty} \delta_n y_n \right) \right|$$

$$= \left| x^* \sum_{n=m+1}^{\infty} (\varepsilon_n - \delta_n) y_n \right| \leq 2 \sum_{n=m+1}^{\infty} |x^*(y_n)| \leq \varepsilon$$

Hence $f(\vec{\delta}) \in f(\vec{\varepsilon}) + W(F, \varepsilon)$. There f is continuous.

By the Riemann-Lebesgue lemma, $F(E) \in c_0 \;\; \forall E$

$$\|F(E)\| = \sup_n \left| \int_E r_n \, d\mu \right| \leq \sup_n \mu(E) = \mu(E)$$

$\uparrow$ Lebesgue measure

Hence

$$|F|(A) = \sup_\pi \sum_{B \in \pi} \|F(B)\| \leq \sup_\pi \sum_{B \in \pi} \mu(B) = \mu(A)$$

and so $|F| << \mu$ and is of bounded variation.

Suppose $\exists$ Bochner integrable $f : [0,1] \to c_0$ s.t.

$$F(E) = \int_E f \, d\mu \quad \forall E$$

Pick $\overset{x^*}{(\beta_n)} \in \ell_1 = c_0^*$. Then

$$x^* F(E) = \int_E x^* f \, d\mu \; ; \; x^* F(E) = \sum_{n=1}^{\infty} \beta_n \int_E r_n \, d\mu$$

Write $f = (\varphi_1, \varphi_2, \ldots)$. Then

$$\int_E x^* f \, d\mu = \sum \beta_n \int_E \varphi_n \, d\mu \quad \forall E \in \Sigma \;\; \forall (\beta_n) \in \ell_1$$

$$\Rightarrow \sum \beta_n \int_E r_n \, d\mu = \sum \beta_n \int_E \varphi_n \, d\mu \quad \forall E \in \Sigma, \forall (\beta_n) \in \ell_1$$

$$\Rightarrow \int_E r_n \, d\mu = \int_E \varphi_n \, d\mu \quad \forall E$$

$$\Longrightarrow \varphi_n = r_n \text{ a.e.}$$

But this means $f = (r_1, r_2, \dots)$ a.e., and hence f has <u>no</u> values in c_0 $\lightning$

2/8 VECTOR MEASURES

[Showing $f(\vec{\varepsilon}) = \sum \varepsilon_n y_n$ is continuous depends on fact that $\sum |x^* y_n| < \infty \ \forall x^*$]

Example: A vector measure of bounded variation absolutely continuous wrt Lebesgue measure on $[0,1]$ that is not an indefinite Bochner integral.

Define $F: \underset{\substack{\uparrow \\ \text{Borel sets in } [0,1]}}{\Sigma} \longrightarrow L_1[0,1]$ by $F(E) = \chi_E$

$$\sum_{E \in \pi} \| F(E) \| = \sum_{E \in \pi} \mu(E) = \mu[0,1] = 1 \qquad \forall \pi$$

Hence F is of bounded variation. Similarly

$$|F|(A) = \mu(A)$$

and so $|F| \ll \mu$.

Suppose $\exists$ Bochner integrable $f: [0,1] \longrightarrow L_1[0,1]$ s.t.

$$F(E) = \int_E f \, d\mu$$

Set

$$\varphi(t) = F([0,t]) = \int_0^t f \, d\mu = \chi_{[0,t]}$$

Just as in 441, $\varphi' = f$ a.e. But φ is nowhere differentiable! (HW) $\square$

For an arbitrary non-atomic measure space (Ω, Σ, μ) define

$$F(E) = \chi_E \in L_1(\mu)$$

Then $F \ll \mu$ but $\{F(E) : E \in \Sigma\}$ is not relatively compact:

E_1 $\qquad \mu(E_1) = \frac{1}{2}\mu(\Omega)$

E_2 $\qquad \mu(E_2) = \frac{1}{2}\mu(\Omega)$

$\mu(E_1 \cap E_2) = \frac{1}{4}\mu(\Omega)$

$\vdots$

Then $\|\chi_{E_n} - \chi_{E_m}\|_{L_1} = \frac{1}{2}\mu(\Omega) \Rightarrow \|F(E_n) - F(E_m)\| = \frac{1}{2}\mu(\Omega)$

Hence F has no derivative

WEAK INTEGRALS

Suppose $f : \Omega \to X$ is weakly measurable and $x^* f \in L_1(\mu)$ for all $x^* \in X^*$. Define $T : X^* \to L_1(\mu)$ by $T(x^*) = x^* f$. Then T is linear

thm
* FACT: T is continuous

Use the closed graph theorem. Suppose $x_n^* \to x^*$ and $Tx_n^* \to g \in L_1(\mu)$

Want to show $T(x^*) = g$. Select a subsequence $(x^*_{n_j})$ of (x^*_n) s.t.

$$T(x^*_{n_j}) = x^*_{n_j} f \longrightarrow g \quad \text{a.e.}$$

But $x^*_n \to x^* \Rightarrow x^*_{n_j} f \to x^* f$ pointwise. Hence

$$g = x^* f = T(x^*)$$

Consider $T^* : L_\infty(\mu) \longrightarrow X^{**}$

$$\langle x^*, T^*(\chi_E) \rangle = \langle T(x^*), \chi_E \rangle$$

$$= \int_\Omega x^* f \chi_E \, d\mu$$

$$= \int_E x^* f \, d\mu$$

Hence for each $E \in \Sigma$, there exists $x^{**}_E \in X^{**}$ s.t.

$$x^{**}_E(x^*) = \int_E x^* f \, d\mu \quad \left[x^{**}_E = D\text{-}\int_E f \, d\mu \right]$$

This is called the Dunford integral of f over E

Radon measure space - (Ω, Σ, μ) Ω = compact Hausdorff space, Σ = Borels, μ regular measure

Conjectured 1935, solved 1977

THEOREM: (Stegall) If (Ω, Σ, μ) is a Radon measure space and $f : \Omega \to X$ is bounded Dunford integrable (ie. $x^*f \in L_1(\mu)$), then

$$R = \left\{ D\text{-}\int_E f \, d\mu : E \in \Sigma \right\}$$

is relatively norm compact

Proof. Enough to show the operator T defined earlier is a compact operator. Why? Because then $T^* : L_\infty \to X^{**}$ is a compact operator (Schauder's Thm) and

$$R = \{ T^*(\chi_E) : E \in \Sigma \}$$

which is T^* of a bounded set in L_∞ and hence relatively compact set

Suppose T is not compact. Then $\exists \, (x_n^*) \subset X^*$ with $\|x_n^*\| \le 1$ s.t. $T(x_n^*) = x_n^* f$ has no pointwise convergent subsequence (since $x_n^* f$ ptwise convergent $\Rightarrow L_1$ convergent since $|x_n^* f| \le \|f\| \le M$ so can use bounded or dominated convergence theorem)

By a very heavy theorem of Fremlin's (Manuscript math 1975) there is a subseq. $(x_{n_i}^* f)$ of $(x_n^* f)$ s.t. no pointwise cluster point of $(x_{n_i}^* f)$ is measurable. Take a w^*-convergent subnet (x_α^*) of $(x_{n_i}^*)$. Let $x_\alpha^* \to x^*$ (weak*)

unit ball is w^* compact

$$x_\alpha^* f \longrightarrow x^* f \quad \text{ptwise}$$

Since f is Dunford-integrable, $x^* f$ is measurable. However, by Fremlin's theorem it is not measurable ↯. ∎

<u>DEFINITION:</u> If $f:\Omega\to X$ is Dunford integrable and

$$D-\int_E f\,d\mu\in X \quad \forall E\in\Sigma$$

then we say f is Pettis integrable.

<u>PROPOSITION:</u> If $f:\Omega\to X$ is Pettis integrable, then $P-\int_E f\,d\mu$ is a countably additive function of E

Proof. Let (E_n) be disjoint seq. in Σ. Let (E_{n_j}) be any subseq. Let $x^*\in X^*$

$$x^*\left(P-\int_{\bigcup_{j=1}^{\infty}E_{n_j}} f\,d\mu\right)=\int_{\bigcup_{j=1}^{\infty}E_j} x^*f\,d\mu=\sum_{j=1}^{\infty}\int_{E_{n_j}}x^*f\,d\mu$$

$$=\sum_{j=1}^{\infty}x^*\left(P-\int_{E_{n_j}}f\,d\mu\right)$$

Hence every subseries of $\sum_{n=1}^{\infty}P-\int_{E_{n_j}}f\,d\mu$ is weakly convergent. By the Orlicz-Pettis theorem $\sum_{n=1}^{\infty}P-\int_{E_n}f\,d\mu$ is norm convergent. By the argument above $x^*\left(P-\int_{\bigcup E_n}f\,d\mu\right)=x^*\left(\sum_{n=1}^{\infty}P-\int_{E_n}f\,d\mu\right)$ $\forall x^*$, so

$$P-\int_{\bigcup E_n}f\,d\mu=\sum_{n=1}^{\infty}P-\int_{E_n}f\,d\mu.$$

HW/ Let $f:\Omega \to X$ be Pettis integrable and let $T: X \to Y$ be a bounded linear operator. Show

$$T\left(P-\int_E f\,d\mu\right) = P-\int_E Tf\,d\mu$$

Example: Let $f:[0,1] \longrightarrow c_0$ be defined by

$$f(t) = \left(\chi_{(0,1)}(t),\, 2\chi_{(0,1/2)}(t),\, 3\chi_{(0,1/3)}(t), \ldots\right).$$

Take $x^* = (\alpha_n) \in \ell_1 = c_0^*$ and notice

$$x^*f = \sum_{n=1}^{\infty} \alpha_n n \chi_{(0,1/n)}$$

so that

$$\int_0^1 |x^*f|\,d\mu \le \sum_{n=1}^{\infty} |\alpha_n| n \cdot \frac{1}{n} = \sum_{n=1}^{\infty} |\alpha_n| < \infty$$

Hence f is Dunford integrable

Also

$$\int_0^1 x^*f\,d\mu = \sum_{n=1}^{\infty} \alpha_n$$

$$\Longrightarrow D-\int_{[0,1]} f\,d\mu = (1,1,1,\ldots) \in \ell_\infty \setminus c_0$$

Therefore f is not Pettis integrable.

Note that

$$D-\int_0^{1/n} f\,d\mu = (*,*,*,\dots,*,1,1,1,\dots)$$

n^{th} place

Hence $\|D-\int_0^{1/n} f\,d\mu\|_{\ell_\infty} \geq 1 \ \forall n$, so that $D-\int_E f\,d\mu$ is not a countably additive function of E (sets $(0,1/n)\downarrow 0$)

Operators on $L_1(\mu)$ and the Radon-Nikodym Property

Riesz Representation "Theorem": Every bounded linear operator $T: L_1(\mu)\to X$ is of the form

$$T(f) = \int_\Omega fg\,d\mu$$

for some Bochner integrable $g:\Omega\to X$

Radon-Nikodym "Theorem": Every countably additive vector measure $F:\Sigma\to X$ that is of bounded variation and μ-continuous is of the form

$$F(E) = \int_E f\,d\mu$$

for some Bochner integrable $f:\Omega\to X$ and all $E\in\Sigma$.

Examples:

① Failure of Riesz representation Thm

Let $T : L_1(\mu) \to L_1(\mu)$ (μ non-atomic) be the identity.
Suppose $\exists g : \Omega \to L_1(\mu)$ Bochner integrable s.t.

$$T(f) = \int_\Omega f g \, d\mu \quad \forall f \in L_1(\mu)$$

Then put $F(E) = T(\chi_E)$. Then $\forall E \in \Sigma$

$$\chi_E = F(E) = T(\chi_E) = \int \chi_E g \, d\mu = \int_E g \, d\mu$$

But the first example today shows this is impossible

DEFINITION: $T : L_1(\mu) \to X$ is representable if T obeys the Riesz representation theorem. g is called the kernel of T

DEFINITION: A Banach space X has the Radon-Nikodym property with respect to (Ω, Σ, μ) if the Radon-Nikodym theorem is true for X

The space X has RNP if it has RNP w.r.t. all finite measure spaces.

Proposition: $L_1(\mu)$ (μ - non-atomic) and c_0 both fail RNP

Proof. See examples

Lemma: Let $T : L_1(\mu) \longrightarrow X$ be a bounded linear operator. For $E \in \Sigma$ define

$$G(E) := T(\chi_E)$$

Then T is representable if and only if there exists a Bochner integrable g s.t.

$$G(E) = \int_E g \, d\mu \quad \forall E \in \Sigma$$

In this case

$$T(f) = \int_\Omega f g \, d\mu \quad \forall f \in L_1(\mu)$$

and $\text{ess.sup} \|g\|_X = \|g\|_\infty = \|T\|$.

Proof. Suppose T is representable. $\exists$ Bochner integrable g s.t

$$T(f) = \int f g \, d\mu \quad \forall f \in L_1(\mu)$$

$$\Rightarrow G(E) = T(\chi_E) = \int_\Omega \chi_E g \, d\mu = \int_E g \, d\mu \quad \forall E \in \Sigma$$

Notice Conversely, suppose $\exists g$ s.t. $T(\chi_E) = \int_E g\,d\mu = G(E)$

$$\|G(E)\| = \|T(\chi_E)\| \leq \|T\|\,\|\chi_E\|_{L_1} = \|T\|\,\mu(E)$$

It follows that $|G|(E) \leq \|T\|\,\mu(E)$. Hence

$$\int_E \|g\|\,d\mu = |G|(E) \leq \|T\|\,\mu(E)$$

$$\Rightarrow \|g\|_X \leq \|T\| \text{ a.e.} \Rightarrow \text{ess.sup}\,\|g\|_X \leq \|T\|$$

On the other hand $T(f) = \int fg\,d\mu$ for all simple functions f. Therefore if f is simple

$$\|T(f)\| = \left\|\int fg\,d\mu\right\| \leq \int \|f\|\,\|g\|\,d\mu \leq \|f\|_{L_1}\,\text{esssup}\,\|g\|_X$$

2/13 VECTOR MEASURES

<u>THEOREM</u>: The space X has RNP w.r.t. (Ω, Σ, μ) if and only if every operator $T: L_1(\mu) \longrightarrow X$ is representable.

Proof. ($\Longrightarrow$) If X has RNP w.r.t. (Ω, Σ, μ) and $T: L_1(\mu) \to X$ is a bounded linear operator, then $G(E) := T(\chi_E)$ is a μ-continuous measure of bounded variation. By RNP there is a Bochner integrable $g: \Omega \to X$ s.t.

$$T(\chi_E) = G(E) = \int_E g\, d\mu$$

Apply lemma to see that

$$T(f) = \int_\Omega fg\, d\mu \quad \forall f \in L_1(\mu)$$

Hence T is representable.

($\Longleftarrow$) Let $G: \Sigma \to X$ be a μ-continuous measure of bounded variation. The Hahn decomposition theorem (or otherwise) produces disjoint sets (E_n) in Σ s.t.

$$(n-1)\mu(E \cap E_n) \leq |G|(E \cap E_n) \leq n\mu(E \cap E_n) \quad \forall n$$

for all $E \in \Sigma$.

(Otherwise: Write $|G|(E) = \int_E \varphi\, d\mu$ and put $E_n = [n-1 \leq \varphi \leq n]$)

Define $T_n: L_1(\mu) \longrightarrow X$ by

$$T_n\left(\sum_{i=1}^{k}\alpha_i \chi_{A_i}\right) = \sum_{i=1}^{k}\alpha_i G(A_i \cap E_n)$$

Then

$$\left\| T_n\left(\sum \alpha_i \chi_{A_i}\right)\right\| \leq \sum |\alpha_i| |G|(A_i \cap E_n)$$

$$\leq \sum |\alpha_i| \, n\mu(A_i \cap E_n)$$

$$\leq n \sum |\alpha_i| \mu(A_i) = n \left\| \sum \alpha_i \chi_{A_i} \right\|_1$$

Hence $T_n : L_1(\mu) \longrightarrow X$ is bounded.

By hypothesis there is a Bochner integrable g_n s.t.

$$T_n(f) = \int_\Omega f g_n \, d\mu \quad \forall f \in L_1(\mu)$$

(WLOG g_n vanishes off E_n.)

In addition

$$G(E \cap E_n) = T_n(\chi_E \chi_{E_n}) = \int_\Omega \chi_E \chi_{E_n} g_n \, d\mu$$

Hence if

$$g(\omega) = \begin{cases} g_n(\omega) & \omega \in E_n \\ 0 & \text{otherwise} \end{cases}$$

then

$$G(E) = \lim_m G\left(E \cap \left(\bigcup_{k=1}^{m} E_k\right)\right) = \lim_m \int_E g \chi_{\bigcup_{k=1}^{m} E_k} \, d\mu$$

But

$$|G|(E) \geq |G|\left(E \cap \bigcup_{k=1}^{m} E_k\right) = \int_E \|g\| \chi_{\bigcup_{k=1}^{m} E_k} \, d\mu$$

By monotone convergence, $\|g\| \in L_1(\mu)$ and since g is "obviously" measurable, we see that g is Bochner-integrable. By dominated convergence with dominator $\|g\|$, we see that

$$G(E) = \int_E g \, d\mu$$

Hence X has RNP w.r.t. (Ω, Σ, μ)

$\boxed{/}$

<u>**DEFINITION**</u>: (Dunford-Morse) A basis (x_n) of a B-space X is called boundedly complete if for all sequences (α_n) of scalars

$$\sup_k \left\| \sum_{n=1}^{k} \alpha_n x_n \right\| < \infty \implies \sum_{n=1}^{\infty} \alpha_n x_n \text{ norm convergent}$$

Examples:

① Boundedly complete basis:

Let $e_n := (0, 0, \ldots, 0, 1, 0, \ldots)$ unit vector in ℓ_p, $1 \leq p < \infty$

Then

Open question: X RNP $\overset{?}{\Rightarrow}$ $\exists$ a subspace Y of X s.t. Y has a boundedly complete basis

$$\sup_k \left\| \sum_{n=1}^{k} \alpha_n e_n \right\| < \infty \Rightarrow \sup_k \left(\sum_{n=1}^{k} |\alpha_n|^p \right)^{1/p} < \infty$$

$$\Rightarrow (\alpha_n) \in \ell_p \underset{1 \leq p < \infty}{\Rightarrow} \sum_{n=1}^{k} \alpha_n e_n \xrightarrow{\ell_p} (\alpha_n)$$

② Not boundedly complete

Let e_n be as above but in c_0. Then $\sup_k \left\| \sum_{n=1}^{k} e_n \right\|_{c_0} = 1$ but $\sum e_n$ does not converge in c_0.

THEOREM: (Dunford) A Banach space X with a boundedly complete basis has RNP.

Proof. Let X have a boundedly complete basis (x_n) with the property

$$\left\| \sum_{n=1}^{k} \alpha_n x_n \right\| \leq \left\| \sum_{n=1}^{k+1} \alpha_n x_n \right\| \qquad \forall (\alpha_n) \subset \mathbb{R}$$

Let x_n^* be the coordinate functionals for x_n, i.e.

$$x = \sum_{n=1}^{\infty} x_n^*(x) x_n$$

Let $G : \Sigma \rightarrow X$ be a μ-continuous vector measure of bounded variation. Then for $E \in \Sigma$,

$$G(E) = \sum_{n=1}^{\infty} x_n^* G(E) x_n$$

Each $x_n^* G(E)$ is countably additive and μ-continuous because x_n^* is continuous and G is countably additive and μ-continuous. Hence $\forall n$ there exists $g_n \in L_1(\mu)$ and

$$x_n^* G(E) = \int_E g_n \, d\mu$$

(Note: μ-continuous vector measure $\Longrightarrow$ countably additive since μ countably additive)

Hopefully $g = \sum g_n x_n$ converges pointwise in X to derivative of G. To prove this, notice

$$\left\| \sum_{n=1}^{k} g_n x_n \right\|_X \leq \left\| \sum_{n=1}^{k+p} g_n x_n \right\|$$

Also observe

$$\left\| \int_E \sum_{n=1}^{k} g_n x_n \, d\mu \right\| = \left\| \sum_{n=1}^{k} (x_n^* G(E)) x_n \right\|$$

$$\leq \left\| \sum_{n=1}^{\infty} (x_n^* G(E)) x_n \right\| = \| G(E) \|$$

$$\leq |G|(E) < \infty$$

Therefore

$$\int_\Omega \left\| \sum_{n=1}^{k} g_n x_n \right\| d\mu \leq |G|(\Omega)$$

But $\left\| \sum_{n=1}^{k} g_n x_n \right\| \uparrow$ as $k \uparrow$. By monotone convergence, there exists an $L_1(\mu)$ function φ s.t.

$$\left\| \sum_{n=1}^{k} g_n x_n \right\| \uparrow \varphi \text{ a.e.}$$

By defining property of boundedly complete bases, $\sum g_n x_n$ is a.e. convergent to a measurable $g : \Omega \to X$

Claim: g is Bochner integrable

Notice

$$\int_\Omega \|g\| d\mu \leq \underline{\lim_k} \int_\Omega \left\| \sum g_n x_n \right\| d\mu \leq |G|(\Omega)$$

Fatou

Now by the dominated convergence theorem with dominator $\|g\| (= \varphi)$, we have

$$G(E) = \lim_k \sum_{n=1}^{k} x_n^* G(E) x_n = \lim_k \int \sum_{n=1}^{k} g_n x_n \, d\mu = \int_E g \, d\mu.$$

☒

__Corollary__: Neither $L_1[0,1]$ nor c_0 has a boundedly complete basis.

__Corollary__: If X has a boundedly complete basis, then every $T: L_1(\mu) \longrightarrow X$ is representable.

__Theorem__ (Lewis-Stegall 1975) The space X has RNP w.r.t. (Ω, Σ, μ) if and only if each $T: L_1(\mu) \longrightarrow X$ admits a factorization

$$\begin{tikzcd} L_1(\mu) \arrow[rr, "T"] \arrow[dr, "S"'] & & X \\ & \ell_1 \arrow[ur, "L"'] & \end{tikzcd} \qquad T = LS$$

where L, S are bounded linear operators. In this case we can choose $\|L\| \le 1$ and $\|S\| \le \|T\| + \varepsilon$ $\quad (S = S(\varepsilon),\ L = L(\varepsilon))$

Proof. We shall prove T is representable $\Leftrightarrow$ it admits the indicated factorization

Suppose T admits the factorization. Since ℓ_1 has RNP (since it has a boundedly complete basis), the operator $S: L_1(\mu) \longrightarrow \ell_1$ is representable. Hence $\exists$ Bochner integrable $h: \Omega \longrightarrow \ell_1$ s.t.

$$S(f) = \int f h \, d\mu \qquad \forall f \in L_1(\mu)$$

Then $\forall f \in L_1(\mu)$

$$T(f) = LS(f) = L\left(\int_\Omega fh\,d\mu\right) = \int_\Omega f(L(h))\,d\mu$$

kernel Bochner integrable

Hence T is representable.

Now suppose T is representable

$$T(f) = \int fg\,d\mu$$

for some Bochner integrable $g: \Omega \to X$. Then

$$\|T\| = \|g\|_\infty = \text{ess.sup.}\, \|g\|_X \quad \left(\text{WLOG } \|g\|_X \le \|T\| \text{ everywhere}\right)$$

With the help of a corollary to Pettis measurability theorem, select (after $\varepsilon > 0$ is set), a seq. (f_n) of countably valued functions s.t. WLOG

$$\|g - f_n\|_X < \varepsilon 2^{-n-1}$$

Put $g_1 = f_1$ and $g_n = f_n - f_{n-1}$ for $n \ge 2$. Write

$$g_n = \sum_{k=1}^{\infty} x_{n,k} \chi_{E_{n,k}}$$

disjoint for n fixed, positive measure

Then $\|x_{1,k}\| \le \|T\| + \varepsilon/2$ since $\|g\|_\infty = \|T\|$ and $\|g - f_1\| = \|g - g_1\| < \varepsilon/2$, and for $n \ge 2$ $\|x_{n,k}\| < \varepsilon 2^{-n}$

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$$\|x_{1,k}\| = \|\underbrace{g_1(w)}_{w \in E_{1,k}}\| \leq \|g(w)\| + \|g(w) - g_1(w)\| \leq \|T\| + \varepsilon/4$$

$$\|x_{n,k}\| = \|g_n(w)\| = \|f_n(w) - f_{n-1}(w)\|$$

$$\leq \|g(w) - f_n(w)\| + \|g(w) - f_{n-1}(w)\| \leq \frac{\varepsilon}{2^{n+1}} + \frac{\varepsilon}{2^{n-1}} = \frac{3\varepsilon}{2^{n+1}}$$

Define $S : L_1(\mu) \longrightarrow \ell_1(\mathbb{N}\times\mathbb{N})$ $(= \ell_1(\mathbb{N}))$ by

$$S(f)(n,k) := \|x_{n,k}\| \int_{E_{n,k}} f\,d\mu$$

Then

$$\|S(f)\|_{\ell_1} \leq \sum_{k=1}^{\infty} \|x_{1,k}\| \int_{E_{1,k}} |f|\,d\mu + \sum_{n=2}^{\infty}\sum_{k=1}^{\infty} \|x_{n,k}\| \int_{E_{n,k}} |f|\,d\mu$$

$$\leq (\|T\| + \varepsilon/4)\|f\|_1 + 3\sum_{n=2}^{\infty} \varepsilon 2^{-n-1}\|f\|_1$$

$$\leq (\|T\| + \varepsilon)\|f\|_1$$

and so $\|S\| \leq \|T\| + \varepsilon$.

Define $L : \ell_1(\mathbb{N}\times\mathbb{N}) \longrightarrow X$ by

$$L((\alpha_{n,k})) = \sum_{n,k} \alpha_{n,k}\, x_{n,k}/\|x_{n,k}\|$$

This series converges since ~~the $(x_{n,k})$ are uniformly bounded and~~ $\sum |\alpha_{n,k}| < \infty$

Now for $f \in L_1(\mu)$

$$LS(f) = L\left(\|x_{n,k}\| \int_{E_{n,k}} f\,d\mu\right)$$

$$= \sum_{n,k} \|x_{n,k}\| \int_{E_{n,k}} f\,d\mu \,\frac{x_{n,k}}{\|x_{n,k}\|}$$

$$= \sum_{n,k} \left(\int_{E_{n,k}} f \, d\mu \right) x_{n,k}$$

$$= \sum_{n=1}^{\infty} \int f g_n \, d\mu$$

$$= \int_{\Omega} f g \, d\mu = T(f)$$

$\boxtimes$

<u>Known</u> (Lewis - Stegall) If X has RNP and is a complemented subspace of $L_1[0,1]$, then X is isomorphic to ℓ_1

<u>Open</u> Let X be a complemented subspace of $L_1[0,1]$ that fails RNP. Must $L_1[0,1]$ be isomorphic to a subspace of X?

☆ prob!
Unit ball of subspace of L_1 compact in measure $\Rightarrow$ subspace is not reflexive unless finite dimensional

2/15 VECTOR MEASURES

$L_1(\mu, X)$ is the space of X-valued Bochner integrable functions with norm

$$\|f\|_1 = \int_\Omega \|f\| \, d\mu$$

$L_\infty(\mu, X) \subseteq L_1(\mu, X)$ with norm

$$\|f\|_\infty = \text{ess.sup} \, \|f\|$$

$K_\infty(\mu, X)$ = subspace of $L_\infty(\mu, X)$ consisting of functions with an essentially relatively compact range.

HW/ Show ∃ infinite dimensional space X & non trivial μ (non-atomic?) s.t. simple functions are not dense in $L_\infty(\mu, X)$. Show $K_\infty(\mu, X)$ is closed (opt)

On the other hand, simple functions are dense in $K_\infty(\mu, X)$. In fact, $K_\infty(\mu, X)$ = closure in $L_\infty(\mu, X)$ of the simple functions. To prove this, notice any simple function is in $K_\infty(\mu, X)$. Since can easily be shown that $K_\infty(\mu, X)$ is closed in $L_\infty(\mu, X)$, we see that closure of simple functions $\subseteq K_\infty(\mu, X)$. Conversely, let $f \in K_\infty(\mu, X)$. WLOG $f(\Omega)$ is relatively compact and hence totally bounded. Put

$$f_\varepsilon = \sum_{i=1}^{n} \underbrace{x_i}_{\varepsilon\text{-net elements}} \chi_{f^{-1}(B_\varepsilon(x_i))}$$

Then $\|f - f_\varepsilon\|_X < \varepsilon$ everywhere, so $\|f - f_\varepsilon\|_\infty < \varepsilon$

For each partition π and $f \in L_1(\mu, X)$, define

$$E_\pi(f) = \sum_{A \in \pi} \frac{\int_A f \, d\mu}{\mu(A)} \chi_A$$

$\left(\frac{0}{0} = 0\right)$

Facts about E_π

① $E_\pi : L_1(\mu, X) \to L_1(\mu, X)$ is a contraction
② $E_\pi : L_\infty(\mu, X) \to L_\infty(\mu, X)$ is a contraction

Proof of ①:

$$\|E_\pi(f)\|_1 = \int_\Omega \left\| \sum_{A \in \pi} \frac{\int_A f \, d\mu}{\mu(A)} \chi_A \right\| d\mu$$

$$= \sum_{A \in \pi} \left\| \int_A f \, d\mu \right\|$$

$$\leq \sum_{A \in \pi} \int_A \|f\| \, d\mu = \int_\Omega \|f\| \, d\mu = \|f\|_1$$

Proof of ②: Let $f \in L_\infty(\mu, X)$. Notice

$$\left\| \frac{\int_A f \, d\mu}{\mu(A)} \right\| \leq \frac{\int_A \|f\| \, d\mu}{\mu(A)} \leq \text{ess sup} \|f\|_X = \|f\|_\infty$$

Then

$$\|E_\pi(f)\|_\infty = \max_{A \in \pi} \left\| \frac{\int_A f \, d\mu}{\mu(A)} \right\|_X \leq \|f\|_\infty$$

③ If $f \in L_1(\mu, X)$, then

$$\lim_\pi \|E_\pi(f) - f\|_1 = 0$$

Proof Let $\varepsilon > 0$ and g a simple functions s.t. $\|f-g\|_1 < \varepsilon/2$. Choose a partition π_0 s.t.

$$\pi \geq \pi_0 \implies E_\pi(g) = g$$

Then if $\pi \geq \pi_0$,

$$\|f - E_\pi(f)\| \leq \|f-g\|_1 + \|g - E_\pi(f)\|$$

$$= \|f-g\|_1 + \|E_\pi(g-f)\|$$

$$\leq \|f-g\|_1 + \|E_\pi\| \, \|g-f\|_1 < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

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Again $\|E_\pi\| = 1$

$g = \sum_{A \in \pi_0} x_A \chi_A$. If $\pi \geq \pi_0$, then

$$E_\pi(g) = \sum_{A \in \pi_0} \sum_{\substack{B \in \pi \\ \cup B = A}} \frac{\int_B \sum_{C \in \pi_0} x_C \chi_C}{\mu(B)} \chi_B$$

$$= \sum_{A \in \pi_0} \sum_{\substack{B \in \pi \\ \cup B = A}} x_A \chi_B = \sum_{A \in \pi_0} x_A \sum_{\substack{B \in \pi \\ \cup B = A}} \chi_B = \sum_{A \in \pi_0} x_A \chi_A = g$$

(4) Same thing with $L_1(\mu, X)$ replaced by $K_\infty(\mu, X)$

THEOREM (Representation of compact operators on $L_1(\mu)$. $K(L_1(\mu), X)$, the space of compact operators from $L_1(\mu)$ to X, $= K_\infty(\mu, X)$, in the sense that each $T \in K(L_1(\mu), X)$ corresponds to some $g \in K_\infty(\mu, X)$ under the action

$$T(f) = \int_\Omega f g \, d\mu \quad \forall f \in L_1(\mu)$$

$$\|T\| = \|g\|_\infty$$

Proof. Let $f \in L_1(\mu)$, $g \in L_\infty(\mu)$. Let π be a partition. Notice

$$\int_\Omega E_\pi(f) g \, d\mu = \sum_{A \in \pi} \frac{\int_A f \, d\mu \int_A g \, d\mu}{\mu(A)}$$

$$= \int_\Omega f \, E_\pi(g) \, d\mu$$

Hence E_π is self-adjoint.

Let $T : L_1(\mu) \longrightarrow X$ be a compact operator. Then $(T \circ E_\pi)^* : X^* \longrightarrow L_\infty(\mu)$ has the form $E_\pi^* T^* = E_\pi T^*$. Now $T^* : X^* \to L_\infty(\mu)$ is also compact. Also, $E_\pi \varphi \longrightarrow \varphi$ in $L_\infty(\mu)$ $\forall \varphi \in L_\infty(\mu)$. Since $\|E_\pi\| \leq 1$, $E_\pi \varphi \longrightarrow \varphi$ uniformly in $\varphi \in$ any relatively compact set. Since T^* is compact, $\lim_\pi E_\pi T^*(x^*) \longrightarrow T^* x^*$ uniformly in $\|x^*\| \leq 1$

Therefore

$$\|E_\pi T^* - T^*\| = \sup_{\|x^*\| \leq 1} \|E_\pi T^*(x^*) - T^*(x^*)\|_\infty \longrightarrow 0 \text{ as } \pi \to \infty$$

and so

$$\| TE_\pi - T \| = \| E_\pi T^* - T^* \| \longrightarrow 0 \text{ as } \pi \to \infty$$

Now define for each partition π

$$(\bigstar) \qquad g_\pi := \sum_{A\in\pi} \frac{T(\chi_A)}{\mu(A)} \chi_A \in K_\infty(\mu, X)$$

(simple function)

Notice if $f \in L_1(\mu)$, then

$$\int_\Omega f g_\pi \, d\mu = \sum_{A\in\pi} \frac{\int_A f \, d\mu \, T(\chi_A)}{\mu(A)}$$

Also

$$TE_\pi(f) = T\left(\sum_{A\in\pi} \frac{\int_A f \, d\mu}{\mu(A)} \chi_A \right)$$

$$= \sum_{A\in\pi} \frac{\int_A f \, d\mu}{\mu(A)} T(\chi_A)$$

Hence

$$TE_\pi(f) = \int_\Omega f g_\pi \, d\mu$$

Hence

$$TE_{\pi'}(f) - TE_{\pi}(f) = \int_{\Omega} f(g_{\pi'} - g_{\pi})\, d\mu$$

But $\|TE_{\pi'} - TE_{\pi}\| \longrightarrow 0$ and $\|TE_{\pi'} - TE_{\pi}\| = \|g_{\pi'} - g_{\pi}\|$, so $(g_\pi)_{\pi \text{ a partition}}$ is a Cauchy net in $K_\infty(\mu, X)$. $\exists\, g \in K_\infty(\mu, X)$ s.t. $\|g_\pi - g\|_\infty \to 0$

Now put $S(f) := \int fg\, d\mu$, and notice that for $f \in L_1(\mu)$,

$$T(f) = \lim_{\pi} TE_\pi(f) = \lim_{\pi} \int_{\Omega} f g_\pi\, d\mu = \int_{\Omega} fg\, d\mu$$

$$\left\| \int fg_\pi - fg\, d\mu \right\| \leq \int |f|\, \|g - g_\pi\|\, d\mu$$

$$\leq \|f\|_1 \|g - g_\pi\|_\infty$$

Hence T compact $\longrightarrow g \in K_\infty(\mu, X)$, i.e. every compact $T : L_1(\mu) \to X$ is of the advertised form.

To complete the proof let $g \in K_\infty(\mu, X)$ and let

$$T(f) = \int fg\, d\mu \quad f \in L_1(\mu, X)$$

Have to show that T is compact.

Select simple functions (g_n) in $K_\infty(\mu, X)$ s.t.

$\|g_n - g\|_\infty \longrightarrow 0$. Define $T_n : L_1(\mu) \longrightarrow X$ by

$$T_n(f) = \int_\Omega f g_n \, d\mu$$

Since each g_n has a finite range, the "T_n"'s are all finite rank operators. Also

$$\|T(f) - T_n(f)\| = \left\| \int_\Omega f(g - g_n) \, d\mu \right\|$$

$$\leq \|f\|_1 \, \|g - g_n\|_\infty \longrightarrow 0 \text{ unif. in } \|f\|_1 \leq 1$$

Hence $\|T - T_n\| \longrightarrow 0$, so T, as the operator topology limit of finite rank operators, is compact. ▨

<u>COROLLARY</u>: Any compact operator $T : L_1(\mu) \longrightarrow X$ is the operator limit of finite rank operators.

Proof. $\|TE_\pi - T\| \longrightarrow 0$

finite rank

COROLLARY: Let $T : L_1(\mu) \longrightarrow X$. For each $E \in \Sigma$ define

$$T \circ E(f) := T(f \chi_E)$$

Then T is representable iff for each $\varepsilon > 0$ $\exists\, E \in \Sigma$ with $\mu(\Omega \setminus E) < \varepsilon$

such that $T\circ E$ is compact.

($\Longrightarrow$ important)

Proof. Suppose T is representable, i.e.

$$T(f) = \int_\Omega fg\,d\mu \quad \forall f\in L_1(\mu)$$

for some $g\in L_\infty(\mu,X)$ (since $\|g\|_\infty = \|T\| < \infty$). Take a sequence (g_n) of simple functions s.t. $g_n \longrightarrow g$ a.e. By Egoroff's theorem $\|g_n - g\| \to 0$ almost uniformly, i.e. if $\varepsilon > 0$ is given, $\exists\, E\in\Sigma$ with $\mu(\Omega\setminus E) < \varepsilon$ s.t.

$$g_n \chi_E \longrightarrow g\chi_E \quad \text{unif.}$$

Hence $g\chi_E \in K_\infty(\mu,X)$. Hence

$$T\circ E(f) = \int f\, \underbrace{g\chi_E}_{\text{element of } K_\infty(\mu,X)}\, d\mu$$

and so $T\circ E$ is compact

For the converse, use the hypothesis to find an increasing seq. $(E_n)\uparrow\Omega$ s.t. $T\circ E_n$ is compact. Put $A_1 = E_1$, $A_2 = E_2\setminus A_1, \ldots$ Then $T\circ A_i$ is compact $\forall i$, so $\exists\, g_i \in K_\infty(\mu,X)$ s.t.

$$T\circ A_i(f) = \int f g_i\, d\mu$$

Put $g(w) = g_i(w)$ for $w\in A_i$, and observe $T(f) = \int fg\,d\mu$.

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$$\int fg\,d\mu = \sum \int f g_i \chi_{A_i}\, d\mu = \sum T \circ A_i (f\chi_{A_i})$$

$$= \sum T(f\chi_{A_i}) = T\left(\sum f\chi_{A_i} \right) = T(f)$$

<u>DEFINITION:</u> A set $K \subset L_1(\mu)$ is uniformly integrable if

$$\lim_{\mu(A)\to 0} \int_A |f|\,d\mu = 0$$

uniformly in $f \in K$

Fact – A bounded set $K \subset L_1(\mu)$ is uniformly integrable iff

$$\lim_{n\to\infty} \int_{[|f|>n]} |f|\,d\mu = 0$$

uniformly in $f \in K$.

<u>THEOREM:</u> A representable operator $T: L_1(\mu) \to X$ takes bounded uniformly integrable sets into norm compact sets.

Proof. Let $\varepsilon > 0$. Choose $\delta > 0$ s.t.

$$\mu(A) < \delta \implies \sup_{f\in K} \int_A |f|\,d\mu < \varepsilon$$

(where K is our bounded uniformly integrable set in $L_1(\mu)$)

Pick $E \in \Sigma$ so large that $\mu(\Omega\setminus E) < \delta$ and $T \circ E$ is compact

Then if $f \in K$, we have

$$T(f) = T(f\chi_{\Omega\setminus E} + f\chi_E)$$

$$= T(f\chi_{\Omega\setminus E}) + T\circ E(f)$$

Now $\{T\circ E(f) : f \in K\}$ is relatively compact since $T\circ E$ is compact and K is bounded. Also

$$\|T\|\,\|f\chi_{\Omega\setminus E}\| \leq \|T\|\varepsilon$$

Hence everything in $T(K)$ is within $\varepsilon\|T\|$ of something in $T\circ E(K)$.
Hence $T(K)$ is totally bounded $\Longrightarrow$ relatively compact

relatively compact set

$\blacksquare$

The reason bounded uniformly integrable sets are of interest is that they are the relatively weakly compact sets in $L_1(\mu)$ [Dunford]. Accepting this we see that representable operators on $L_1(\mu)$ carry relatively weakly compact sets to relatively norm compact sets (weakly convergent seq. are carried to norm convergent seq)

<u>**THEOREM**</u> (Dunford-Pettis) A weakly compact operator $T : L_1(\mu) \to X$ whose range is (norm) separable, is representable

Proof. For each partition π, put

$$g_\pi = \sum_{A\in\pi} \frac{T(\chi_A)}{\mu(A)}\chi_A$$

Notice

$$g_\pi(\Omega) \subset T(\text{unit ball of } L_1) \leq \text{sep. weakly compact set}$$

since $\|\chi_A/\mu(A)\|_1 = 1$

Basic idea: For each $\omega \in \Omega$, let $g(\omega)$ = arb. weak cluster point of $g_\pi(\omega)$. Then $g(\omega) \in W$. Then g has separable range. If we could show $x^* g$ is measurable $\forall x^*$ and justify $x^* g_\pi \to x^* g$, then we'd have g Bochner integrable and $x^* T(f) = \lim x^* T(E_\pi(f))$

$$= \lim_\pi x^* \int f g_\pi \, d\mu = \lim_\pi \int f \, x^* g_\pi \, d\mu = \int f g \, d\mu$$

2/20 VECTOR MEASURES

THEOREM (Dunford-Pettis TAMS 1940) A weakly compact $T : L_1(\mu) \to X$ whose range is separable is representable.

Proof. Put

$$g_\pi = \sum_{A \in \pi} \frac{T(\chi_A)}{\mu(A)} \chi_A$$

Then

$$g_\pi(\Omega) \subseteq \left\{ \frac{T(\chi_E)}{\mu(E)} : E \in \Sigma \right\} \subseteq T(\text{unit ball of } L_1) \subseteq \underset{\text{compact } K}{\text{sep. weakly}}$$

WLOG X is separable (otherwise take $X = \overline{sp}\, K$) Therefore X^* has a countable norming set, i.e. there is a sequence (x_n^*) in X^* with $\|x_n^*\| = 1$ and

$$\sup |x_n^*(x)| = \|x\| \qquad \forall x \in X$$

Notice

$$x_n^* T(f) = \int f g_n \, d\mu \qquad \forall f \in L_1(\mu)$$

for some $g_n \in L_\infty(\mu)$. Notice

$$\int_\Omega f x_n^* g_\pi \, d\mu = \sum_{A\in\pi} \frac{\int_A f\,d\mu \; x_n^*(T(\chi_A))}{\mu(A)} \, d\mu$$

$$= \sum_{A\in\pi} \frac{\int_A f\,d\mu \int_A g_n\,d\mu}{\mu(A)}$$

$$= \int_\Omega E_\pi(f)\, g_n \, d\mu$$

$$= \int_\Omega f\, E_\pi(g_n)\, d\mu \qquad \forall f \in L_1(\mu)$$

Therefore $x_n^* g_\pi = E_\pi(g_n) \quad \forall n \;\forall \pi$

We know $\lim_\pi \|E_\pi(g_n) - g_n\|_\infty = 0$ for each n, so there exists a sequence π_m of partitions s.t.

$$\lim_m \| E_{\pi_m}(g_n) - g_n \|_\infty = 0 \quad \forall n$$

(diagonalization) Therefore

$$\lim_m \| x_n^* g_{\pi_m} - g_n \|_\infty = 0 \quad \forall n$$

Therefore there exists a null set P s.t. for each fixed n

$$x_n^* g_{\pi_m} \xrightarrow{m\to\infty} g_n \text{ unif off } P$$

(not necessarily uniform in n) For each $\omega \notin P$, let $g(\omega)$ be an arbitrary weak cluster point of $(g_{\pi_m}(\omega))$ (values in weakly compact set K). Then g is separably valued.

We claim that g is measurable. We know that for $\omega \notin P$

$$x_n^* g_{\pi_m}(\omega) \xrightarrow{m\to\infty} g_n(\omega)$$

But also

$$x_n^* g_{\pi_m}(\omega) \xrightarrow{m\to\infty} x_n^* g(\omega)$$

for all n, and so $x_n^* g$ is a.e. pointwise equal to the measurable functions g_n. Hence $x_n^* g$ is measurable. By the Pettis measurability theorem g is measurable. Also $g(\Omega) \subseteq K$, so g is bounded, whence $g \in L_\infty(\mu, X)$. Therefore

$$\int_\Omega f g \, d\mu \text{ exists } \forall f \in L_1(\mu)$$

But for each n and $f \in L_1(\mu)$,

$$x_n^* T(f) = \int_\Omega f g_n \, d\mu = \int_\Omega f x_n^* g \, d\mu$$

$$= x_n^* \int_\Omega f g \, d\mu$$

Since (x_n^*) separates points of X, we see that

$$T(f) = \int_\Omega fg\,d\mu$$

$\blacksquare$

<u>THEOREM</u>: (Dunford-Pettis) Separable dual spaces have RNP.

Proof. Consider $T: L_1(\mu) \to X^*$, where X^* is separable. Define g_π as above. Replace the x_n^*'s by x_n's from X. Use weak $*$ compactness of bounded sets in X^* to define g. By the separability of X^*, g has a separable range

mutatis mutandis

$\blacksquare$

<u>COROLLARY</u> (of penultimate theorem) A weakly compact operator on $L_1(\mu)$ has a separable range, and hence any weakly compact $T: L_1(\mu) \to X$ is representable.

Proof. Let $T: L_1(\mu) \to X$ be weakly compact. Notice that the range of $T \subseteq \overline{sp}\{T(\chi_E) : E \in \Sigma\}$ (since simple functions dense in $L_1(\mu)$). To show that T has a separable range, it suffices to prove that $\{T(\chi_E) : E \in \Sigma\}$ is relatively norm compact. To this end, let $(T(\chi_{E_n}))$ be a sequence in this set. ~~Let~~

$$\Sigma_1 := \sigma(E_n : n \in \mathbb{N})$$

Then $L_1(\Sigma_1, \mu|\Sigma_1)$ is a subspace of $L_1(\mu)$, and $L_1(\Sigma_1, \mu|_{\Sigma_1})$ is separable (since Σ_1 is countably generated). $T | L_1(\Sigma_1, \mu|_{\Sigma_1})$ is still weakly compact and has a separable range. Call this new operator T_1. Then T_1 is representable. The set $\{\chi_{E_n} : n \in \mathbb{N}\}$ is uniformly integrable and bounded. Hence $\{T_1(\chi_{E_n}) : n \in \mathbb{N}\} = \{T(\chi_{E_n}) : n \in \mathbb{N}\}$ is relatively compact, and so $(T(\chi_{E_n}))$ has a norm convergent subsequence. This proves $\{T(\chi_E) : E \in \Sigma\}$ is relatively norm compact

$\boxtimes$

__COROLLARY__ (Phillips) All reflexive spaces have RNP

Proof. Let $T : L_1(\mu) \to X$ be a bounded linear operator. Suppose X is reflexive. Then T is a weakly compact operator since the unit ball of X is weakly compact. Therefore T is representable.

$\boxtimes$

__COROLLARY__ (Dunford-Pettis) Weakly compact operators on $L_1(\mu)$ take weakly compact sets into norm compact sets.

Proof. All weakly compact sets in $L_1(\mu)$ are bounded and uniformly integrable

$\boxtimes$

[* Prel. Weakly compact in $Y \Leftrightarrow$ weakly compact in X]
$Y \subset X$

<u>COROLLARY</u>: No infinite dimensional reflexive subspace of $L_1(\mu)$ is complemented.

Proof. Suppose Y is a reflexive subspace of $L_1(\mu)$ complemented by a projection P. Let B denote the unit ball of $L_1(\mu)$. Then $P(B)$ is relatively weakly compact in Y and hence in $L_1(\mu)$.
Since Y is reflexive, P is weakly compact operator. Hence $P(P(B))$ is relatively norm compact. But $P(P(B)) \supseteq P(B)$. By Open Mapping theorem, $P(B)$ contains an open subset of Y. Hence Y is finite dimensional (unit ball is compact)

$\boxtimes$

FACT: A Banach space is reflexive iff all $T: L_1(\mu) \to X$ are weakly compact

Proof. $\Rightarrow$ obvious
$\Leftarrow$ Suppose X is not reflexive. Take a sequence (x_n) in unit ball of X with no weakly convergent subsequence. Let (E_n) be a disjoint sequence of Borel measurable sets each of positive measure. Put

$$g = \sum_{n=1}^{\infty} x_n \chi_{E_n}$$

Define $T: L_1[0,1] \to X$ by

$$T(f) = \int fg \, d\mu$$

Then T is representable, but

$$T(\text{unit ball of } L_1) \ni \left(\frac{T(\chi_{E_n})}{\mu(E_n)} \right) = (x_n)$$

and so T is not weakly compact.

$\boxtimes$

Toward RN Theorems

Exhaustion Lemma: Let (Ω, Σ, μ) be a finite measure space. Let P be a property that a set $E \in \Sigma$ has or fails. Suppose

(1) All null sets have P

(2) A, B have $P \implies A \cup B$ has P

(3) A has P, $B \subseteq A$, $B \in \Sigma \implies B$ has P

(4) If $A \in \Sigma$ and $\mu(A) > 0$, then $\exists B \in \Sigma$ with $\mu(B) > 0$ and $B \subset A$ such that B has P

Then there exists a disjoint sequence (A_n) s.t. each A_n has P and $\Omega = \bigcup_{n=1}^{\infty} A_n$.

Proof. Let $\mathcal{Q} := \{E \in \Sigma : E \text{ has } P\}$. Let

$$c := \sup_{E \in \mathcal{Q}} \mu(E)$$

Let (B_n) be a sequence in $\mathcal{Q}$ such that $\mu(B_n) \to c$. Put

$$E_m = \bigcup_{n=1}^{m} B_n$$

② Then $E_m \in \mathcal{A}$ and $\mu(E_m) \uparrow c$. Suppose $c < \mu(\Omega)$. Then

$$\mu\left(\Omega \setminus \bigcup_{n=1}^{\infty} E_n\right) > 0$$

④ Hence there exists $A \in \mathcal{A}$ s.t. $\mu(A) > 0$ and $A \subset \Omega \setminus \bigcup_{n=1}^{\infty} E_n$. Then $E_n \cup A \in \mathcal{A}$ $\forall n$, and

$$\mu(E_n \cup A) = \mu(E_n) + \mu(A) \to c + \mu(A) > c$$

which contradicts the choice of c. Hence $c = \mu(\Omega)$.
Now put

$$A_0 = \Omega \setminus \bigcup_{n=1}^{\infty} E_n$$

⑦ (null set, so A has P).

$$A_1 = E_1$$

$$\vdots$$

$$A_{k+1} = E_{k+1} \setminus E_k \in \mathcal{A} \quad ③$$

Then (A_n) does the job.

$\boxtimes$

<u>COROLLARY</u>: Let $G : \Sigma \to X$ be a μ-continuous measure of bounded variation with the property that for each $A \in \Sigma$ of positive μ-measure $\exists$ $B \in \Sigma$, $B \subset A$, $\mu(B) > 0$ and a Bochner integrable

g_B supported on B s.t.

$$G(E \cap B) = \int_{E \cap B} g_B \, d\mu$$

$\forall E \in \Sigma$. Then there exists a Bochner integrable g s.t.

$$G(E) = \int_E g \, d\mu$$

$\forall E \in \Sigma$.

Proof. Use exhaustion lemma. Say a set A has property P if $\exists$ a Bochner integrable g_A vanishing off A s.t.

$$G(E \cap A) = \int_{E \cap A} g_A \, d\mu$$

This satisfies (1)-(4). Hence there is a disjoint seq (A_n) whose union is Ω and Bochner integrable g_n vanishing off A_n s.t.

$$G(E \cap A_n) = \int_{E \cap A_n} g_n \, d\mu$$

$\forall E \in \Sigma$. Put

$$g = \sum_{n=1}^{\infty} g_n$$

(g_n's disjointly supported). Then

(*) $$G(E) = \lim_m G\left(E \cap \bigcup_{n=1}^m A_n\right) = \lim_m \int_E g\chi_{\bigcup_{n=1}^m A_n} \, d\mu$$

also

$$G\left(E \cap \bigcup_{n=1}^m A_n\right) = \int_{E \cap \left(\bigcup_{n=1}^m A_n\right)} g \, d\mu \quad \forall E \in \Sigma$$

and so

$$|G|(\Omega) \geq |G|\left(\bigcup_{n=1}^m A_n\right) = \int_{\bigcup_{n=1}^m A_n} \|g\| \, d\mu$$

By monotone convergence $\|g\| \in L_1(\mu)$ and g is measurable, so g is Bochner integrable. Therefore in (*)

$$G(E) = \int_E g \, d\mu$$

2/22 VECTOR MEASURES

RADON - NIKODYM THEOREM: Let $G : \Sigma \to X$ be a μ-continuous bounded variation vector measure. Then if for each set $A \in \Sigma$ of positive μ-measure, there exists $B \in \Sigma$, $\mu(B) > 0$ and $B \subset A$ such that

$$\left\{ \frac{G(E)}{\mu(E)} : E \subseteq B \right\}$$

is relatively weakly compact (Average range of G is locally relatively weakly compact), then there exist a Bochner integrable g s.t.

$$G(E) = \int_E g \, d\mu$$

$\forall E \in \Sigma$.

Conversely, if there is a Bochner integrable g s.t.

$$G(E) = \int_E g \, d\mu$$

then for each $\varepsilon > 0$ $\exists E \in \Sigma$ with $\mu(\Omega \setminus E) < \varepsilon$ s.t.

$$\left\{ \frac{G(A)}{\mu(A)} : A \subset E \right\}$$

is relatively norm compact.

Proof. For the first part it is enough to prove that for each $A \in \Sigma$ with $\mu(A) > 0$ there exists $B \in \Sigma$ with $\mu(B) > 0$ and $B \subseteq A$, and Bochner integrable g_B s.t.

$$G(E \cap B) = \int_{E \cap B} g_B \, d\mu \quad \forall E \in \Sigma$$

To this end, choose B as guaranteed in hypothesis, i.e.

$$K = \left\{ \frac{G(E)}{\mu(E)} : E \subseteq B \right\}$$

is relatively weakly compact.

Define $T : L_1(B \mid \mu|_B) \longrightarrow X$ by

$$T\left(\sum \alpha_i \chi_{E_i} \right) = \sum \alpha_i \mu(E_i) \frac{G(E_i)}{\mu(E_i)}$$

Now if $\sum |\alpha_i| \mu(E_i) = \| \sum \alpha_i \chi_{E_i} \| \leq 1$, then

$$T\left(\sum \alpha_i \chi_{E_i} \right) \in \underbrace{\overline{co}\left(K \cup \{0\} \cup -K \right)}_{\text{weakly compact by Krein-Smulian}}$$

Hence T extends to a weakly compact operator from $L_1(\mu|_B)$ into X.
Since T is weakly compact, it is representable, i.e. $\exists\, g_B$ s.t. supported on B

$$T(f) = \int_B f g_B \, d\mu \quad \forall f \in L_1(\mu|_B)$$

Accordingly,

$$G(E \cap B) = T\left(\chi_{E \cap B} \right) = \int_{E \cap B} g_B \, d\mu \quad \forall E \in \Sigma$$

For part 2, suppose

$$G(E) = \int_E g\,d\mu \quad \forall E \in \Sigma$$

for some Bochner integrable g.

Choose sequence (g_n) of simple functions s.t. $g_n \to g$ a.e.

Let $\varepsilon > 0$. Choose $E \in \Sigma$ s.t. $\mu(\Omega \setminus E) < \varepsilon$ and $g_n \to g$ uniformly on E.

Notice $g\chi_E \in K_\infty(\mu, X)$. Therefore

$$T(f) := \int_E fg\,d\mu$$

defines a compact operator from $L_1(\mu)$ into X

If $B \subset E$, then

$$T\left(\frac{\chi_B}{\mu(B)}\right) = \int_E \frac{\chi_B}{\mu(B)} g\,d\mu = \frac{G(B)}{\mu(B)}$$

Therefore

$$\left\{ \frac{G(B)}{\mu(B)} : B \subset E \right\} \subseteq T(\text{unit ball of } L_1(\mu))$$

relatively compact set

$\blacksquare$

HW/ Let G be a μ-cont. bdd variation vector measure. Then G is an indefinite Bochner integral iff $\forall \varepsilon > 0$ and A of positive measure $\exists\, B \subseteq A$, $\mu(B) > 0$ s.t. $\operatorname{diam}\left\{ G(E)/\mu(E) : E \subset B \right\} < \varepsilon$.

Opt HW/ Let G be μ-cont, bdd variation. Suppose $\left(G(E_n)/\mu(E_n)\right)$ is a relatively compact seq. $\forall$ disjoint (E_n). Then G is an indefinite Bochner integral

[Can not replace E by Ω since not every representable operator is compact (weakly compact)]

FACT: Let $T : L_1(\mu) \to X$ be a bounded linear operator. Define for $E \in \Sigma$ $T \circ E$ by $T \circ E(f) = T(f \chi_E)$. TFAE

1. T is representable
2. $\forall \varepsilon > 0$ $\exists E$ with $\mu(\Omega \setminus E) < \varepsilon$ s.t. $T \circ E$ is weakly compact
3. $\forall \varepsilon > 0$ $\exists E$ with $\mu(\Omega \setminus E) < \varepsilon$ s.t. $T \circ E$ is norm compact
4. $\forall A \in \Sigma$ positive measure, $\exists E \subset A$ $\mu(E) > 0$ s.t. $T \circ E$ weakly compact
5. " " " " " " " norm compact

THEOREM: A Banach space has RNP if each of its (closed) separable subspaces has RNP.

If a Banach space has RNP, then each of its closed subspaces has RNP

Proof. For the first statement, let $T : L_1(\mu) \to X$ be a bounded linear operator. If we can show that T is representable, then we'll be done. If we can show that T has a separable range, then we will have shown that T is representable (since then T will be into an RNP space since all separable subspaces of X have RNP). To do this it is sufficient to show $\{T(\chi_E) : E \in \Sigma\}$ is relatively compact ($\Rightarrow$ separable). Let $(T(\chi_{E_n}))$ be a seq. in this set and let $\Sigma_1 = \sigma(\{E_n\})$. Define for $f \in L_1(\Sigma_1)$, $T_1(f) = T(f)$. Since $L_1(\Sigma_1)$ is separable, $T_1 : L_1(\Sigma_1) \to X$ has separable range, and so T_1 is representable. Therefore

$$\{T(\chi_{E_n})\} = \{T_1(\chi_{E_n})\}$$

is relatively compact, and so $T(\chi_{E_n})$ has a convergent subsequence.

To prove the second statement, suppose X has RNP and let Y be a subspace of X. Let $G : \Sigma \to Y$ be a μ-continuous vector measure of bounded variation. To show: $\exists\, g : \Omega \to Y$ Bochner integrable s.t.

$$G(E) = \int_E g \, d\mu$$

We know $\exists\, h : \Omega \to X$ such that

$$G(E) = \int_E h \, d\mu$$

We also know $E_\pi(h) \to h$ in $L_1(\mu, X)$. Therefore $\exists$ seq (π_n) of partitions s.t.

$$E_{\pi_n}(h) \to h \text{ a.e.}$$

But

$$E_\pi(h) = \sum_{A\in\pi} \frac{\int_A h\,d\mu}{\mu(A)} \chi_A = \sum_{A\in\pi} \frac{G(A)}{\mu(A)} \chi_A$$

and so $E_\pi(h)$ has its range in Y. Hence h is essentially Y-valued. Take

$$g(\omega) = \begin{cases} h(\omega) & \text{if } h(\omega) \in Y \\ 0 & \text{otherwise} \end{cases}$$

$\boxed{/}$

THEOREM: Let $G : \Sigma \to X$ be a μ-continuous vector measure of bounded variation. Put

$$g_\pi = \sum_{E\in\pi} \frac{G(E)}{\mu(E)} \chi_E$$

Then $\exists\, g \in L_1(\mu, X)$ s.t.

$$G(E) = \int_E g\,d\mu$$

if and only if (g_π) is a Cauchy net in $L_1(\mu, X)$.

COROLLARY: Under the same general hypothesis, $\exists\, g \in L_1(\mu, X)$ s.t.

$$G(E) = \int_E g\,d\mu$$

provided (g_{π_n}) is $L_1(\mu, X)$ Cauchy for all sequences of partitions with $\pi_n \leq \pi_{n+1}$ s.t. $\max\{\mu(A) : A \in \pi_n\} \to 0$ as $n \to \infty$

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If $G(E) = \int_E g\,d\mu$, then

$$g_\pi = \sum_{A\in\pi} \frac{\int_A g\,d\mu}{\mu(A)} \chi_A = E_\pi(g) \longrightarrow g$$

so (g_π) is Cauchy

Conversely, if g_π is Cauchy, let $g = \lim g_\pi$. Then if $E \in \Sigma$ and $\varepsilon > 0$, choose partition π which is a refinement of $\{E, \Omega\setminus E\}$ s.t.

$$\|g - g_\pi\|_{L_1} < \varepsilon$$

Then

$$\int_E g_\pi = G(E)$$

and so

$$\left\| G(E) - \int_E g\,d\mu \right\| = \left\| \int_E g_\pi - g\,d\mu \right\| \leq \int_\Omega \|g_\pi - g\|\,d\mu < \varepsilon$$

Since ε was arbitrary, we see that $G(E) = \int_E g\,d\mu$.

(non-atomic case)

<u>COROLLARY</u>: A Banach space has RNP if and only if it has RNP with respect to Lebesgue measure on $[0,1]$.

Proof. $(\Leftarrow)$ WLOG (Ω, Σ, μ) = non-atomic probability space. Suppose X has RNP w.r.t. Lebesgue measure on $[0,1]$. Let $G : \Sigma \to X$ be μ-continuous bounded variation. Take (π_n) with $\pi_n \leq \pi_{n+1}$ and

$$\lim_{n\to\infty} \max\{\mu(A) : A \in \pi_n\} = 0$$

Take $\pi_1 = \{A_1, \ldots, A_n\}$. Find disjoint half open - half closed intervals in $[0,1)$ $\tilde{A}_1, \ldots, \tilde{A}_n$ s.t. $\mu(A_i) = \lambda(\tilde{A}_i)$ (λ = Leb measure). Take

$$\pi_2 = \{B_1^1, B_2^1, \ldots, B_{n_1}^1, B_1^2, \ldots, B_{n_2}^2, \ldots\}$$

where $A_i = \bigcup_k B_k^i$. Chop the $\tilde{A}_i$'s up into $\tilde{B}_k^i$ in the obvious way. Take

$$g_{\pi_n} \longmapsto \tilde{g}_{\tilde{\pi}_n} \in L_1([0,1), X)$$

Notice Borel sets $= \sigma\left(\bigcup_n \tilde{\pi}_n\right)$. Also notice that

$$\lim_n \int_E \tilde{g}_{\tilde{\pi}_n}\, d\lambda$$

exists $\forall E \in \bigcup_n \tilde{\pi}_n$. It is not hard to show that the limit exists $\forall$ Borel sets and defines a λ-continuous vector measure $\tilde{G}$ on the Borel sets. Since X has RNP w.r.t. λ

$$\tilde{G}(E) = \int_E \tilde{g}\, d\mu$$

Therefore

$$E_{\tilde{\pi}_n}(\tilde{g}) = \tilde{g}_{\pi_n} \longrightarrow \tilde{g} \text{ in } L_1(\mu, X) \text{ norm}$$

But $\|g_{\pi_k} - g_{\pi_{k+\ell}}\|_{L_1(\mu,X)} = \|\tilde{g}_{\tilde{\pi}_k} - \tilde{g}_{\tilde{\pi}_{k+\ell}}\|_{L_1(\lambda,X)} \to 0$ as $k,\ell \to \infty$

Hence (g_{π_k}) is $L_1(\mu, X)$ Cauchy, so G has a derivative.

$\blacksquare$

Consider RNP in $[0,1)$. Let

$$\pi_1 = \{[0,1)\}$$

$$\pi_2 = \{[0,1/2), [1/2, 1)\}$$

$$\vdots$$

dyadic partitions

Since simple functions relative to the dyadic partitions are dense in $L_1(\mu, X)$ we have $E_{\pi_n}(f) \to f$ in $L_1([0,1), X)$ norm $\forall f \in L_1([0,1), X)$

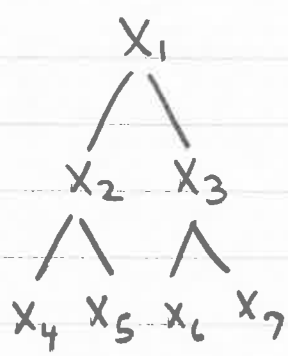
THEOREM: If a Banach space contains a bounded δ-tree then it fails RNP

[A tree in a Banach space X is a sequence (x_n) s.t.

$$x_n = \frac{1}{2}(x_{2n} + x_{2n+1}) \quad \forall n$$

A tree is a bounded δ-tree if $\sup \|x_n\| < \infty$ and

$$\|x_n - x_{2n}\| \geq \delta, \quad \|x_n - x_{2n+1}\| \geq \delta$$]



Proof. Define F : dyadic intervals $\longrightarrow X$ as follows

$$F([0,1]) = x_1 \mu[0,1)$$

$$F([0,1/2)) = x_2 \mu[0,1/2)$$

$$F([1/2,1)) = x_3 \mu[1/2,1)$$

$$F\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right) = x_{2^n+k}$$

$$0 \leq k \leq 2^n - 1$$

(Notice $F([0,1)) = F([0,1/2)) + F([1/2,1))$.)

$$F([0,1/4)) = x_4 \mu[0,1/4)$$

Define T : dyadic simple functions $\longrightarrow X$ by

$$T\left(\sum \alpha_i \chi_{I_i}\right) = \sum \alpha_i F(I_i)$$

dyadic intervals

T is well-defined

$$\left\|T\left(\sum \alpha_i \chi_{I_i}\right)\right\| \le \sum |\alpha_i| \, \|F(I_i)\|$$

$$\le \sum |\alpha_i| \sup_n \|x_n\| \, \mu(I_i)$$

$$= \sup \|x_n\| \, \left\|\sum \alpha_i \chi_{I_i}\right\|_{L_1([0,1])}$$

Hence T is bounded on a dense subset of $L_1([0,1])$ and so has a continuous extension to all of $L_1[0,1]$

Suppose T is representable.

$$T(f) = \int f g \, d\mu$$

Then

$$F(I) = T(\chi_I) = \int_I g \, d\mu$$

Also $E_{\pi_n}(g) \to g$ in $L_1([0,1], X)$. But

$$\|E_{\pi_n} g - E_{\pi_{n+1}} g\| \ge \delta$$

PROPOSITION: Let $T : L_1(\mu) \to X$ be a bounded linear operator. Define for $E \in \Sigma$, $T \circ E(f) := T(f \chi_E)$. Then TFAE

1. T is representable
2. $\forall \varepsilon > 0$ $\exists E$ with $\mu(\Omega \setminus E) < \varepsilon$ s.t. $T \circ E$ is weakly compact
3. $\forall \varepsilon > 0$ $\exists E$ with $\mu(\Omega \setminus E) < \varepsilon$ s.t. $T \circ E$ is norm compact
4. $\forall A \in \Sigma$ of positive measure $\exists E \subset A$, $\mu(E) > 0$, s.t. $T \circ E$ is weakly compact
5. $\forall A \in \Sigma$ of positive measure $\exists E \subset A$, $\mu(E) > 0$, s.t. $T \circ E$ is norm compact

Proof. Suppose (1) holds. Define

$$G(E) := T(\chi_E)$$

Since T is representable, G has a Bochner derivative g. Hence, if $\varepsilon > 0$ there exists $E \in \Sigma$ with $\mu(\Omega \setminus E) < \varepsilon$ such that

$$K := \left\{ \frac{G(A)}{\mu(A)} : A \subset E \right\}$$

is norm compact

Let $f = \sum \alpha_i \chi_{A_i}$, with $\|f\| = \sum |\alpha_i| \mu(A_i) \leq 1$. Then

$$T \circ E \left(\sum \alpha_i \chi_{A_i} \right) = \sum \alpha_i T(\chi_{A_i \cap E}) = \sum \alpha_i G(A_i \cap E)$$

$$= \sum' \alpha_i \mu(A_i \cap E) \frac{G(A_i \cap E)}{\mu(A_i \cap E)}$$

$$\in \overline{co}\left(K \cup -K \cup \{0\} \right)$$

But $K \cup -K \cup \{0\}$ is norm compact, so by Mazur's theorem $\overline{co}(K \cup -K \cup \{0\})$ is also norm compact. Hence

$$T \circ E\left(\text{unit ball of } L_1(\mu)\right) \subset \overline{co}(K \cup -K \cup \{0\})$$

shows that $T \circ E$ is norm compact (and hence also weakly compact). Therefore (2) and (3) hold.

Still supposing that (1) holds, let $A \in \Sigma$ be of positive measure. For the same reason as before, choose $B \in \Sigma$ such that $\mu(\Omega \setminus B) < \mu(A)$ and

$$K = \left\{ \frac{G(F)}{\mu(F)} : F \subset B \right\}$$

is norm compact. Let $E = A \cap B$. Then

$$\mu(\Omega) < \mu(A) + \mu(B) = \mu(A \cup B) - \mu(A \cap B)$$

and so $\mu(A \cap B) > 0$. For the same reason as before, $T \circ E$ is norm compact (and hence also weakly compact). Therefore (4) and (5) hold.

Now suppose (2) holds. Again define

$$G(E) = T(\chi_E)$$

Then G is a μ-continuous vector measure of bounded variation. Let $A \in \Sigma$ be of positive measure. By (2) choose E with $\mu(\Omega \setminus E) < \mu(A)$ and $T \circ E$ weakly compact. If $B = E \cap A$, then $\mu(B) > 0$

and for each $F \subset B$

$$\frac{G(F)}{\mu(F)} = \frac{T(\chi_F)}{\mu(F)} = \frac{T(\chi_F \chi_E)}{\mu(F)} = \frac{T\circ E(\chi_F)}{\mu(F)}$$

$$= T\circ E\left(\frac{\chi_F}{\mu(F)}\right)$$

and so

$$\left\{\frac{G(F)}{\mu(F)} : F \subset B\right\} \subset T\circ E\left(\text{unit ball of } L_1(\mu)\right)$$

Therefore $\{G(F)/\mu(F) : F \subset B\}$ is relatively weakly compact, whence G has a Bochner derivative. Therefore T is representable.

The same proof shows that if $T\circ E$ is norm compact, then $\{G(F)/\mu(F) : F \in B\}$ is relatively norm compact, so also relatively weakly compact, and thus again T is representable.

That (4) and (5) imply (1) follows in exactly the same way.

$\blacksquare$

2/27 VECTOR MEASURES

Given bounded δ-tree (x_n)

$$\|x_n - x_{2n}\| \geq \delta \qquad x_n = \frac{x_{2n} + x_{2n+1}}{2}$$
$$\|x_n - x_{2n+1}\| \geq \delta$$

Take dyadic partitions

$$\Pi_1 = \{ [0,1] \}$$
$$\Pi_2 = \{ [0,1/2) , [1/2, 1) \}$$
$$\vdots$$

Put $f_1 = x_1 \chi_{[0,1)}$, $f_2 = x_2 \chi_{[0,1/2)} + x_3 \chi_{[1/2,1)}$

$$\begin{tikzcd}[column sep=small]
 & & & x_1 \arrow[dl, no head] \arrow[dr, no head] & & & & f_1 \arrow[llll, leftrightarrow] \\
 & & x_2 \arrow[dl, no head] \arrow[d, no head] & & x_3 \arrow[d, no head] \arrow[dr, no head] & & & f_2 \arrow[lll, leftrightarrow] \\
 & x_4 & x_5 & & x_6 & x_6 & & f_3 \arrow[ll, leftrightarrow] \\
 & & & \vdots & & & & \vdots
\end{tikzcd}$$

For each dyadic interval E, set $F(E) = x_E \mu(E)$. Notice

① F is additive on dyadic intervals by averaging property of tree

Define $T : L_1(\mu) \longrightarrow X$ on the dyadic simple functions by

$$T\left(\sum \alpha_i \chi_{I_i}\right) := \sum \alpha_i F(I_i)$$

well-defined because F is additive.

$$\left\| T\left(\sum \alpha_i \chi_{I_i}\right)\right\| \leq \sum |\alpha_i| \mu(I_i) \left\| \frac{F(I_i)}{\mu(I_i)} \right\|$$

$$\leq \underbrace{M}_{\text{bound for tree}} \left\| \sum \alpha_i \chi_{I_i} \right\|_{L_1}$$

element of tree

Therefore T has a bounded extension to all of $L_1[0,1]$.
Now suppose X has RNP and put

$$G(E) = T(\chi_E)$$

for all Borel sets E. Since X has RNP, T is representable, i.e. G has a Bochner derivative $g : [0,1] \to X$
Now $E_{\pi_n}(g) \to g$ in $L_1(\mu, X)$. In particular

$$\int_{[0,1]} \| E_{\pi_{n+1}}(g) - E_{\pi_n}(g) \|_X \, d\mu \to 0 \quad \text{as } n \to \infty$$

But note

$$E_{\pi_n}(g) = \sum_{E \in \pi_n} \frac{G(E)}{\mu(E)} \chi_E = \sum_{E \in \pi_n} \frac{F(E)}{\mu(E)} \chi_E$$

$$= \sum_{E \in \pi_n} x_E \chi_E$$

Write

$$\begin{array}{lccccc}
\pi_n : & I_1 & I_2 & \cdots & & I_{2^{n-1}} \\
 & \wedge & \wedge & & & \wedge \\
\pi_{n+1} & J_{11} \; J_{12} & J_{21} \; J_{22} & \cdots & &
\end{array}$$

Notice if $t \in J_{11}$, then

$$E_{\pi_n} g(t) - E_{\pi_{n+1}}(g)(t) = x_{I_1} - x_{J_{11}}$$

$$\Longrightarrow \quad \| E_{\pi_n} g(t) - E_{\pi_{n+1}} g(t) \| \geq \| x_{I_1} - x_{J_{11}} \| \geq \delta$$

Same argument shows $\| E_{\pi_n} g - E_{\pi_{n+1}} g \|_X \geq \delta$, and so

$$\underline{\lim_n} \int_{[0,1]} \| E_{\pi_n}(g) - E_{\pi_{n+1}}(g) \| d\mu \geq \delta \quad ↯$$

$\boxtimes$

Hence RNP $\Rightarrow$ no bounded δ - trees $\Rightarrow$ δ trees must grow.

EXAMPLE : Bounded δ - tree in c_0

$x_1 = (0, 0, 0, \ldots)$ $\qquad$ $x_4 = (0, 1, 1, 0, \ldots)$

$x_2 = (0, 1, 0, \ldots)$ $\qquad$ $x_5 = (0, 1, -1, 0, \ldots)$

$x_3 = (0, -1, 0, \ldots)$ $\qquad$ $x_6 = (0, -1, 1, 0, \ldots)$

$x_7 = (0, -1, -1, 0, \ldots)$

Bounded δ- tree in $L_1[0,1]$

$$x_1 = \chi_{[0,1]}$$

$$x_2 = \frac{\chi_{[0,1/2]}}{\mu[0,1/2]} \qquad x_3 = \frac{\chi_{[1/2,1]}}{\mu[1/2,1]}$$

keep splitting

COROLLARY: Reflexive spaces and separable duals do not have bounded δ- trees

COROLLARY: Neither $L_1[0,1]$ nor c_0 are isomorphic to dual spaces

Proof. Both are separable spaces without RNP but spaces isomorphic to separable duals have RNP

COROLLARY: If every separable subspace of X is a copy of a subspace of a separable dual space, then X has RNP

Proof. By hypothesis every separable subspace of X has RNP.

COROLLARY: $\ell_1(\Gamma)$ for any Γ has RNP

Proof. Any separable subspace of $\ell_1(\Gamma)$ is supported on at most countably many γ's. Therefore any separable subspace of $\ell_1(\Gamma)$ is a copy of a subspace of $\ell_1 = c_0^*$.

COROLLARY: Every separable subspace of X has a separable dual implies X^* has RNP

Proof. Let M be a separable subspace of X^*. We shall show that M is isometric to a subspace of a separable dual.

Take a dense seq. (x_n^*) of M. Select sequences $(x_{m,n})_{m=1}^\infty$ in X s.t. $\|x_{m,n}\| = 1$ and

$$|x_n^*(x_{m,n})| \geq \left(1 - \frac{1}{m}\right)\|x_n^*\|$$

Let $Y = \overline{sp}(x_{m,n} : m,n \in \mathbb{N})$. Y is separable. The hypothesis says that Y^* is separable. Define $T: M \to Y^*$ by

$$T(x^*)(y) := x^*(y)$$

Obviously $\|T\| \leq 1$. Also

$$\|T(x_m^*)\| \geq \sup_n |x_m^*(x_{m,n})| \geq \|x_m^*\|$$

Hence T is an isometry because (x_m^*) is dense in M. $\boxtimes$

COROLLARY: WCG dual spaces have RNP

Proof. Recall X is WCG if it is the closed linear span of a weakly compact subset. (Reflexive spaces and separable spaces are both WCG (for separable spaces take $x_n / \|x_n\| n$))

To prove the corollary, we shall show that if X is such that X^* is WCG, then every separable subspace of X has a separable dual. Let Y be a separable subspace of X. Then

$$Y^* = X^* / Y^\perp$$

and $X^*/Y^\perp$ is WCG because X^* is WCG (take image of weakly compact set in X^* under quotient map) Let W be a weakly compact subset of Y^* that generates Y^*

$$Y^* = \overline{sp}\,(W)$$

Notice that W is w^*-compact. Since Y is separable, the w^*-topology on W is a metric topology. Since W is w^*-compact, W is w^*-separable. Since W is also weakly compact, and w^* is weaker than weak, the weak* and weak topology agree on W. Hence W is weakly separable $\Longrightarrow$ norm separable. Therefore

$$Y^* = \overline{sp}\,(W)$$

is also norm separable.

$\square$

(Subspaces of WCG need not be WCG)

<u>**COROLLARY**</u>: X^* a subspace of WCG space $\Rightarrow$ X^* has RNP

Proof. Let M be a separable subspace of X. Want to show M^* is separable. Choose a mapping $T : \ell_1 \longrightarrow M$ (onto) (every separable subspace is quotient of ℓ_1)

$$M^* = X^*/M^\perp$$

isomorphism into since T onto

$$\begin{tikzcd} X^* \arrow[r, "Q"] \arrow[d, "I"'] & M^* \arrow[r, "T^*"] & \ell_\infty \\ Y \arrow[urr, dashed, "S"] & & \end{tikzcd}$$

(WCG)

To find S use Hahn-Banach coordinatewise

Now the weakly compact sets in ℓ_∞ are norm separable (ℓ_∞ is dual of a separable space). Therefore $S(Y)$ is norm separable ($S(Y)$ is WCG since Y is WCG) Therefore $T^*(M^*)$ is separable Therefore M^* is separable since T^* is an isomorphism. ▨

Example: $\ell_1(\Gamma)$ is a dual space with RNP which is not <u>WCG</u> (For uncountable Γ)

(Since $\ell_1(\Gamma)$ has the Schur property, the weakly compact sets = norm compact sets by Eberlein-Smulian. Hence $\ell_1(\Gamma)$, if were WCG, would be the closed linear span of a separable set, and hence would be separable But $\ell_1(\Gamma)$ is not separable if Γ is uncountable)

MARTINGALES

Let (Ω, Σ, μ) be as usual and $\mathcal{B}$ a sub-σ-field of Σ. Take $f \in L_1(\mu)$. Define λ on $\mathcal{B}$ by

$$\lambda(E) = \int_E f \, d\mu$$

Notice $\lambda \ll \mu|_{\mathcal{B}}$. Therefore there exists $\underset{\text{unique}}{\wedge}$ $g \in L_1(\mathcal{B}, \mu|_{\mathcal{B}})$ (by ordinary RN theorem) s.t.

$$\lambda(E) = \int_E g \, d\mu \quad \forall E \in \mathcal{B}$$

We write $E(f|\mathcal{B}) = g$ (conditional expectation of f given $\mathcal{B}$)

Properties of $E(\cdot|\mathcal{B})$

① linear (by uniqueness of RN derivatives)
② contraction on $L_1(\mu)$

$$\int_\Omega |g| \, d\mu = \underset{\text{on } B}{\operatorname{var}} \int_{(\cdot)} g \, d\mu = \underset{\text{on } B}{\operatorname{var}} \int_{(\cdot)} f \, d\mu \leq \underset{\text{on } \Sigma}{\operatorname{var}} \int_{(\cdot)} f \, d\mu = \int_\Omega |f| \, d\mu$$

③ contraction on $L_p(\mu)$ $1 \leq p < \infty$

follows from Jensen's inequality

3/1 VECTOR MEASURES

THEOREM: $E(\cdot|\mathcal{B})$ is a contractive projection on L_p

To get $E(\cdot|\mathcal{B})$ defined on $L_p(\mu, X)$ we define $E(\cdot|\mathcal{B})$ on simple functions by

conditional expectation in $\mathbb{R}$

$$E\Big(\underbrace{\sum x_i \chi_{E_i}}_{\text{disjoint}} \Big| \mathcal{B}\Big) = \sum_{i=1}^{n} x_i E(\chi_{E_i}|\mathcal{B})$$

Notice

$$\left(\int_\Omega \| E(\sum x_i \chi_{E_i} |\mathcal{B})\|^p d\mu\right)^{1/p} = \left(\int_\Omega \|\sum x_i E(\chi_{E_i}|\mathcal{B})\|^p d\mu\right)^{1/p}$$

$$\leq \left(\int_\Omega \left(\sum \|x_i\| E(\chi_{E_i}|\mathcal{B})\right)^p d\mu\right)^{1/p}$$

$$= \left\| E\left(\sum \|x_i\| \chi_{E_i} |\mathcal{B}\right)\right\|_p$$

$$\leq \left\| \sum \|x_i\| \chi_{E_i}\right\|_p = \left\|\sum x_i \chi_{E_i}\right\|_{L_p(\mu, X)}$$

Therefore $E(\cdot|\mathcal{B})$ has a contractive extension to all of $L_p(\mu, X)$ $(1 \leq p < \infty)$ because simple functions are dense in $L_p(\mu, X)$.

It is easily checked that $E(f|\mathcal{B}) = g$ if and only if g is $\mathcal{B}$ measurable and

$$\text{Bochner-}\int_E f\,d\mu = \int_E g\,d\mu \quad \forall E \in \mathcal{B}$$

DEFINITION: Let $(\mathcal{B}_\tau : \tau \in T)$ be an increasing net of sub-σ-fields of $\mathcal{S}$. Let $(f_\tau : \tau \in T)$ be a net of functions such that f_τ is $\mathcal{B}_\tau$-measurable for all τ. Then $(f_\tau, \mathcal{B}_\tau : \tau \in T)$ is called a martingale in $L_p(\mu, X)$ if

① each $f_\tau \in L_p(\mu, X)$

② $E(f_\tau | \mathcal{B}_{\tau_0}) \overset{a.e.}{=} f_{\tau_0}$ for all $\tau \geq \tau_0$

Examples

① Fix $f \in L_p(\mu, X)$ and define $f_\tau := E(f | \mathcal{B}_\tau)$

Each f_τ is obviously $\mathcal{B}_\tau$-measurable. Let $\tau \geq \tau_0$. We know f_{τ_0} is the unique $\mathcal{B}_{\tau_0}$ measurable function s.t.

$$\int_A f \, d\mu = \int_A f_{\tau_0} \, d\mu \quad \forall A \in \mathcal{B}_{\tau_0}$$

Also

$$\int_A f \, d\mu = \int_A f_\tau \, d\mu \quad \forall A \in \mathcal{B}_\tau$$

Hence

$$\int_A f_\tau \, d\mu = \int_A f_{\tau_0} \, d\mu \quad \forall A \in \mathcal{B}_{\tau_0} \subset \mathcal{B}_\tau$$

Therefore $E(f_\tau | \mathcal{B}_{\tau_0}) = f_{\tau_0}$

② Let $F : \Sigma \longrightarrow X$ be any μ-continuous vector measure. Put

$$f_\pi = \sum_{A \in \pi} \frac{F(A)}{\mu(A)} \chi_A$$

for all partitions π. Then $(f_\pi, \sigma(\pi) : \pi \text{ a partition})$ is a martingale. Take $\pi_1 \le \pi_2$. If $A \in \pi_1$, then

$$\int_A f_{\pi_2}\, d\mu = F(A) = \int_A f_{\pi_1}\, d\mu$$

and so $E(f_{\pi_2} \mid \sigma(\pi_1)) = f_{\pi_1}$

③ Let $(\Omega, \Sigma, \mu) = [0,1)$ with Lebesgue measure. Let (x_n) be a tree in X. Write

$$f_1 = x_1 \chi_{[0,1)}$$

$$f_2 = x_2 \chi_{[0,1/2)} + x_3 \chi_{[1/2,1)}$$

$$f_3 = x_4 \chi_{[0,1/4)} + x_5 \chi_{[1/4,1/2)} + x_6 \chi_{[1/2,3/4)} + x_7 \chi_{[3/4,1)}$$

Let $\mathcal{B}_1 = \sigma([0,1))$, $\mathcal{B}_2 = \sigma([0,1/2), [1/2,1))$, etc. The averaging property of trees shows that we obtain a martingale

Essential property of martingales

Let $(f_\tau, \mathcal{B}_\tau)$ be a martingale in $L_p(\mu, X)$. Take $A \in \bigcup_\tau \mathcal{B}_\tau$. Choose τ_0 s.t. $A \in \mathcal{B}_{\tau_0}$. Note that $A \in \mathcal{B}_\tau$ $\forall \tau \geq \tau_0$. Also observe that

$$\int_A f_\tau \, d\mu = \int_A f_{\tau_0} \, d\mu \qquad \forall \tau \geq \tau_0$$

Therefore

$$\lim_\tau \int_A f_\tau \, d\mu = F(A) \text{ exists}$$

limit measure

$\forall A \in \bigcup \mathcal{B}_\tau$

THEOREM: $(1 \leq p < \infty)$ A martingale in $L_p(\mu, X)$ converges (in $L_p(\mu,X)$ norm) iff its limit measure has R-N derivative in $L_p(\mu, X)$, i.e. $\exists f \in L_p(\mu, X)$ s.t.

$$F(A) = \int_A f \, d\mu$$

$\forall A \in \bigcup \mathcal{B}_\tau$

Remark: This is false in $L_\infty(\mu, \mathbb{R})$. Take a function f in $L_\infty(\mu, \mathbb{R})$ that is not L_∞-limit of step functions. Take (Π_n) to be the sequence of dyadic partitions. Put

$$f_n = \sum_{A \in \Pi_n} \frac{\int_A f \, d\mu}{\mu(A)} \chi_A \qquad \text{(martingale)}$$

and suppose $\lim f_n = g$ exists in $L_\infty(\mu, \mathbb{R})$ norm.
Then if $A \in \pi_n$

$$\int_A f \, d\mu = \int_A f_{n_0} d\mu \xrightarrow[n_0 \to \infty]{} \int g \, d\mu$$

Hence $\int_A f \, d\mu = \int_A g \, d\mu$ $\forall$ dyadic intervals $\Rightarrow f = g$ a.e. ↯

Proof of theorem. ($\Rightarrow$) Suppose $\lim f_\tau = f$ exists in $L_p(\mu, X)$ norm
Then

$$\int_A f_\tau \, d\mu \longrightarrow \int_A f \, d\mu \quad \forall A \in \Sigma$$

Therefore

$$F(A) = \lim_\tau \int_A f_\tau \, d\mu = \int_A f \, d\mu$$

$\forall A \in \bigcup \mathcal{B}_\tau$

($\Leftarrow$) Suppose the limit measure has a R-N derivative f, i.e.

$$\lim_\tau \int_A f_\tau \, d\mu = F(A) = \int_A f \, d\mu \quad \forall A \in \bigcup \mathcal{B}_\tau$$

By the martingale property

$$\int_A f_{\tau_0} d\mu = \int_A f \, d\mu \quad \forall A \in \mathcal{B}_{\tau_0}$$

Therefore $E(f|\mathcal{B}_{\tau_0}) = f_{\tau_0}$. If necessary replace f by $E(f|\sigma(\cup\mathcal{B}_\tau))$
Let $\varepsilon>0$. Since f is $\sigma(\cup\mathcal{B}_\tau)$-measurable and $\cup\mathcal{B}_\tau$ is a field of sets, there exist a $\cup\mathcal{B}_\tau$-simple function f_ε s.t.

$$\|f - f_\varepsilon\|_p < \varepsilon/2$$

Pick τ_0 s.t. f_ε is $\mathcal{B}_{\tau_0}$-measurable, i.e.

$$E(f_\varepsilon|\mathcal{B}_\tau) = f_\varepsilon \quad \forall \tau \geq \tau_0$$

Hence

$$\begin{aligned} \tau \geq \tau_0 \Rightarrow \|f - f_\tau\|_p &\leq \|f - f_\varepsilon\|_p + \|f_\varepsilon - f_\tau\|_p \\ &= \|f - f_\varepsilon\|_p + \|E((f_\varepsilon - f)|\mathcal{B}_\tau)\|_p \\ &\leq \|f - f_\varepsilon\|_p + \|f_\varepsilon - f\|_p \leq \varepsilon \end{aligned}$$

$\blacksquare$

<u>COROLLARY</u>: $(f_\tau, \mathcal{B}_\tau : \tau \in T)$ is $L_p(\mu, X)$ convergent $\Leftrightarrow$
$\exists f \in L_p(\mu, X)$ s.t. $E(f|\mathcal{B}_\tau) = f_\tau$

Proof. Look at proof above

DEFINITION: A martingale $(f_\tau, \mathcal{B}_\tau : \tau \in T)$ is called uniformly integrable if

$$\lim_{\substack{\mu(E)\to 0 \\ E\in\mathcal{B}_\tau}} \int_E \|f_\tau\| \, d\mu = 0$$

uniformly in τ

THEOREM: Let X have RNP. Then $L_1(\mu, X)$ bounded uniformly integrable martingales converge in $L_1(\mu, X)$ norm

Proof. Let $(f_\tau, \mathcal{B}_\tau)$ be such a martingale. Then the limit measure is μ-continuous (from uniform integrability). Also the limit measure is of bounded variation. Let π be a partition Ω into $\bigcup \mathcal{B}_\tau$-sets. Let F be the limit measure.

$$\sum_{E\in\pi} \|F(E)\| = \sum_{E\in\pi} \left\| \int_E f_{\tau_0} \, d\mu \right\| \quad \text{where } \tau_0 \text{ is s.t. } \pi \subset \mathcal{B}_{\tau_0}$$

$$\leq \sum_{E\in\pi} \int_E \|f_{\tau_0}\| \, d\mu$$

$$= \|f_{\tau_0}\|_1 \leq K \quad (\text{by } L_1\text{-bdd hypothesis})$$

By something in Chapter 1, the limit measure has a bounded variation μ-continuous extension to $\sigma(\bigcup \mathcal{B}_\tau)$. By RNP, $\exists f \in L_1(\mu, X)$ s.t.

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$F(E) = \lim_{\tau} \int_E g_\tau \, d\mu \quad \forall E \in \bigcup \mathcal{B}_\tau$ (field)

Let $\varepsilon > 0$ and choose δ s.t. $\mu(E) < \delta$, $E \in \mathcal{B}_\tau \overset{(*)}{\Longrightarrow} \int_E \| g_\tau \| d\mu < \varepsilon \quad \forall \tau$

Let $E \in \bigcup \mathcal{B}_\tau$, $\mu(E) < \delta$. Then $E \in \mathcal{B}_{\tau_0}$ for some τ_0 and

$F(E) = \int_E g_{\tau_0} d\mu$. Hence by $(*)$ $\| F(E) \| < \varepsilon$.

Or go through operators

$$F(E) = \int_E f \, d\mu$$

Hence $f_\tau \to f$ in $L_1(\mu, X)$ norm.

<u>COROLLARY</u>: Let $1 < p < \infty$ and X have RNP. Then $L_p(\mu, X)$ bounded martingales converge in $L_p(\mu, X)$ norm.

Proof $1 < p < \infty$ + L_p bdd + Holder inequality $\Rightarrow$ uniformly integrable

$$\left\| \int_E f_\tau \, d\mu \right\| = \left\| \int_\Omega f_\tau \chi_E \, d\mu \right\| \leq \| f_\tau \|_p \, \| \chi_E \|_q \to 0$$

uniformly in τ as $\mu(E) \to 0$

Thus $\lim f_\tau = f$ in L_1-norm. If we can show $f \in L_p(\mu, X)$, then we'll have shown limit measure has an $L_p(\mu, X)$ derivative, and we'll be done. To this end, take τ_n s.t. $f_{\tau_n} \to f$ a.e. Then

$$\int \|f\|^p \, d\mu \leq \varliminf_n \int \|f_{\tau_n}\|^p \, d\mu \leq (L_p\text{- bd for martingale})^p$$

Hence $f \in L_p(\mu, X)$

$\boxtimes$

3/6 VECTOR MEASURES

THEOREM: $L_1(\mu, X)$ bounded uniformly integrable martingales converge in $L_1(\mu, X)$ norm implies X has RNP.

Proof. Let $F: \Sigma \longrightarrow X$ be μ-continuous and bounded variation. Let

$$f_\pi = \sum_{E \in \pi} \frac{F(E)}{\mu(E)} \chi_E$$

This is a martingale

$$F \ll \mu \implies (f_\pi) \text{ uniformly integrable}$$
$$F \text{ bdd var.} \implies (f_\pi) \ L_1\text{-bdd}$$

Hence $f_\pi \longrightarrow f$ in $L_1(\mu, X)$ norm and f is a derivative of F ▨

LEMMA (Maximal lemma) Let $(f_n, \mathcal{B}_n)$ be a martingale in $L_1(\mu, X)$. Let $\delta > 0$ and put

$$S_\delta := \{ \omega : \sup_n \|f_n(\omega)\|_X > \delta \}$$

Then

$$\varlimsup_n \int_{S_\delta} (\|f_n\| - \delta) \, d\mu \geq 0$$

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Note $\sigma(\pi)$ = set of all finite unions of sets in π (since π is finite & disjoint sets)

$\mathcal{B}_\pi = \sigma(\pi)$ $\quad F \ll \mu \implies |F| \ll \mu$. $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\mu(E) < \delta \Rightarrow |F|(E) < \varepsilon$

If $\underset{\mu(E)<\delta}{E \in \mathcal{B}_\pi}$, then E is the finite union of sets in π, i.e. $E = \bigcup_{A \in \tau} A$, $\tau \subset \pi$

$$\int_E \|f_\pi\| \, d\mu = \sum_{A \in \tau} \int_A \|f_\pi\| \, d\mu \le \sum_{A \in \tau} |F|(A) = |F|(\cup A) = |F|(E) < \varepsilon$$

$$\|f_\pi\|_{L_1} = \int \|f_\pi\| \, d\mu = \int \sum_{E \in \pi} \frac{\|F(E)\|}{\mu(E)} \chi_E \, d\mu = \sum_{E \in \pi} \|F(E)\| \le |F|(\Omega)$$

Consequently

$$\mu(S_\delta) \leq \frac{1}{\delta} \sup_n \|f_n\|_1$$

Proof. To prove the last statement from the penultimate one, select $n_1 < n_2 < \dots$ s.t.

$$(?) \quad \int_{S_\delta} (\|f_{n_i}\| - \delta)\, d\mu \geq 0$$

Then

$$\sup_n \|f_n\|_1 \geq \int_{S_\delta} \|f_{n_i}\| \geq \delta \mu(S_\delta)$$

Hold up on rest of proof

need $\mathbb{N}$, not general index set

COROLLARY: An $L_1(\mu, X)$ convergent martingale $(f_n, \mathcal{B}_n)$ converges a.e.

Proof. Let $(f_n, \mathcal{B}_n)$ be $L_1(\mu, X)$ convergent martingale. Let $\varepsilon, \delta > 0$ and pick n_0 s.t.

$$m, n \geq n_0 \implies \|f_n - f_m\|_1 < \varepsilon\delta$$

Fix $m \geq n_0$. Consider $(f_n - f_m, \mathcal{B}_n : n \geq m)$. This is a martingale. By the lemma

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Select $n_1 < n_2 < \cdots$ s.t. $\int_{S_\delta} (\|f_{n_i}\| - \delta)\, d\mu \geq -\frac{1}{i}$. Then

$$\sup_n \| \|f_n\| \|_1 \geq \sup_i \int_{S_\delta} \|f_{n_i}\| \, d\mu \geq \sup_i \left(\delta\mu(S_\delta) - \frac{1}{i} \right) = \delta\mu(S_\delta)$$

$$\mu\left(\{\omega : \sup_{n\geq m} \|f_n(\omega) - f_m(\omega)\|_X > \delta\}\right) < \frac{1}{\delta}\sup_n \|f_n - f_m\|_1 < \varepsilon$$

It follows quickly that (f_n) is almost uniformly Cauchy and hence a.e. convergent.

$\blacksquare$

Proof of maximal lemma (cont.)

Put $S_\delta^m := [\|f_m\|_X > \delta,\ \|f_j\|_X \leq \delta \text{ for } j<m]$. Notice

$$S_\delta = \bigcup_m S_\delta^m$$

$$S_\delta^m \cap S_\delta^n = \phi \quad n \neq m$$

$$S_\delta^m \in \mathcal{B}_m$$

Now

S_δ^n's are disjoint with union S_δ

$$\overline{\lim_n} \int_{S_\delta} (\|f_n\| - \delta)\, d\mu = \overline{\lim_n} \lim_k \sum_{m=1}^{k} \int_{S_\delta^m} (\|f_n\| - \delta)\, d\mu$$

$E(\cdot|\mathcal{B}_m)$ is a contraction on L_1

$$\geq \overline{\lim_n} \lim_k \sum_{m=1}^{k} \int_{S_\delta^m} (\|f_m\| - \delta)\, d\mu \geq 0$$

(fix k - take large enough n)

$\blacksquare$

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Let $\varepsilon > 0$. $\forall n$ $\exists m_n$ s.t. $\mu(\overbrace{\{\omega : \sup_{k \geq m_n} \|f_k(\omega) - f_{m_n}(\omega)\| > 1/2^n\}}^{\Omega \setminus E_n}) < \varepsilon/2^n$

Let $E = \bigcap E_n$. Then $\mu(\Omega\setminus E) = \mu(\bigcup \Omega \setminus E_n) < \varepsilon$. Let $\delta > 0$ and choose n_0 st. $1/2^{n_0} < \delta/2$. If $n, k > m_{n_0}$, then if $\omega \in E$, $\omega \in E_{n_0} \Rightarrow$

$$\|f_n(\omega) - f_k(\omega)\| \leq \|f_n(\omega) - f_{n_0}(\omega)\| + \|f_k(\omega) - f_{n_0}(\omega)\| < 1/2^{n_0} + 1/2^{n_0} < \delta$$

Hence $(f_n \chi_E(\omega))$ is uniformly Cauchy.

Hence $\forall n$ $\exists E_n$ st. $\mu(\Omega \setminus E_n) < 1/2^n$ + $(f_m \chi_{E_n})_{m=1}^{\infty}$ is uniformly Cauchy in X. Let $F = \{\omega : \omega$ is in infinitely many $\Omega\setminus E_n\}$. Then $\mu(F) = 0$. If $\omega \notin F$, then $\exists n_0$ st. $\omega \notin E_{n_0}$. Then $(f_m(\omega)) = (f_m \chi_{E_{n_0}}(\omega))$ is Cauchy in X, so $f_m(\omega)$ converges. Since $f_m \to f$ in $L_1(\mu, X)$, we must have $f_m(\omega) \to f(\omega)$. Hence f_m converges to f a.e.

$$\delta\mu\left(\bigcup_{m=1}^{n} S_\delta^m\right) = \delta \sum_{m=1}^{n} \mu(S_\delta^m) = \sum_{m=1}^{n} \int_{S_\delta^m} \delta \, d\mu \leq \sum_{m=1}^{n} \int_{S_\delta^m} \|f_m\| d\mu$$

$$\leq \sum_{m=1}^{n} \int_{S_\delta^m} \|f_n\| d\mu = \int_{\bigcup_{m=1}^{n} S_\delta^m} \|f_n\| d\mu \leq \int_{S_\delta} \|f_n\| d\mu$$

Hence $\delta\mu(S_\delta) = \overline{\lim_n}\, \delta\mu\left(\bigcup_{m=1}^{n} S_\delta^m\right) \leq \overline{\lim_n} \int_{S_\delta} \|f_n\| d\mu$

$$\Rightarrow \overline{\lim} \int_{S_\delta} (\|f_n\| - \delta) \, d\mu \geq 0$$

HW/ (Metivier) Let $(f_n, \mathcal{B}_n)$ be an $L_1(\mu, X)$ bounded martingale and uniformly integrable s.t. $\forall \omega$ $\exists$ weakly compact set $K_\omega \subset X$ s.t. $f_n(\omega) \in K_\omega$ $\forall n$. Then f_n converges in $L_1(\mu, X)$ norm. Hint $(x^* f_n)$ is a real martingale.

Tree interpretation: Let (x_n) be a bounded tree in X. Each point of $\{+1, -1\}^{\mathbb{N}}$ determines a path through the tree. Let λ = Haar measure. Then X RNP $\Rightarrow$ convergence along almost every path

BEGINNING OF RNP GEOMETRY (or How to stop X from having RNP)

Suppose we produce a martingale $(f_n, \mathcal{B}_n)$ in $L_1([0,1], X)$ s.t

Heuristic

1) $\sup_n \|f_n\|_\infty < \infty$ $\quad$ ($\Rightarrow L_1$-bdd + unif. integ.)

2) each f_n is countably valued

3) $\|f_n(t) - f_{n+1}(t)\| \geq \varepsilon$ for some $\varepsilon > 0$ and all n, t

Then X does not have RNP.

What does this mean? There exists a sequence (Δ_n) of (countable) partitions on Ω s.t. $\mathcal{B}_n = \sigma(\Delta_n)$ and such that

$$A \in \Delta_n \Rightarrow A = \bigcup_{\substack{E \in \Delta_{n+1} \\ E \subset A}} E \qquad (\Delta_n \leq \Delta_{n+1})$$

Write

$$f_n = \sum_{E \in \Delta_n} x_E \chi_E$$

Then ① forces $D := \{ x_E : E \in \Delta_n, n \in \mathbb{N} \}$ to be bounded. ③ forces

$$\|x_E - x_A\| \geq \varepsilon \quad \forall A \in \Delta_n,\ E \in \Delta_{n+1},\ E \subset A$$

The martingale property says

$$\int_A f_{n+1}\, d\mu = \int_A f_n\, d\mu \quad \forall A \in \Delta_n$$

$$\sum_{\substack{E \in \Delta_{n+1} \\ E \subset A}} x_E \mu(E) = x_A \mu(A)$$

i.e.

$$x_A = \sum_{\substack{E \in \Delta_{n+1} \\ E \subset A}} \frac{\mu(E)}{\mu(A)} x_E$$

infinite convex sum

Hence each $x \in D$ is the infinite convex sum of other things in D that are at least ε away from x (generalization of a tree)

DEFINITION: A set D in X is not σ-dentable if there exists $\varepsilon>0$ such that for each $x\in D$ there exists $(\alpha_n(x))$ with $\alpha_n(x)>0$

$$\sum_{n=1}^{\infty} \alpha_n(x) = 1$$

and a sequence $(x_n(x))$ in D such that $\|x - x_n(x)\| \geq \varepsilon$ and

$$x = \sum_{n=1}^{\infty} \alpha_n(x)\, x_n(x)$$

DEFINITION: A set D in X is not dentable if $\exists\, \varepsilon>0$ s.t.

$$x\in D \implies x \in \overline{co}\,(D\setminus B_\varepsilon(x))$$

Notice: Let $D = \{\text{unit vector basis in } \ell_2\} \cup \{0\}$. Note

$$0 \in \overline{co}\,(D\setminus B_{1/4}(0))$$ (unit vector basis converges weakly to 0)

but 0 is not of the form $\sum \alpha_n e_n$.

FACT: The closed unit ball of $L_\infty[0,1]$ is σ-dentable but not dentable

Example: No bounded infinite δ-tree is σ-dentable or dentable

THEOREM: (Maynard) Suppose X has a bounded non-σ-dentable subset D. Then there exist an $L_\infty(\mu, X)$ bounded martingale $(f_n, \mathcal{B}_n)$ of countably valued functions with values in D on $[0,1)$ and there exist $\delta > 0$ s.t.

$$\| f_n(t) - f_{n+1}(t) \|_X \geq \delta$$

$\forall n, \forall t \in [0,1)$. Hence bounded non-$\sigma$-dentable set implies no RNP.

Proof. Pick $\delta > 0$ s.t.

$$x \in D \Rightarrow x = \sum_{n=1}^{\infty} \alpha_n(x)\, x_n(x)$$

Pick $\bar{x} \in D$ arbitrarily. Put $f_1 := \bar{x}\, \chi_{[0,1)}$ and $\mathcal{B}_1 = \{\emptyset, [0,1)\}$. Then

$$f_1 = \sum_{n=1}^{\infty} \alpha_n(\bar{x})\, x_n(\bar{x})\, \chi_{[0,1)}$$

Set $\beta_0 = 0$, $\beta_m = \sum_{n=1}^{m} \alpha_n(\bar{x})$. Write

$$I_m = [\beta_{m-1}, \beta_m) \quad [\mu(I_m) = \alpha_m(\bar{x})]$$

Define

$$f_2 = \sum_{m=1}^{\infty} x_m(\bar{x})\, \chi_{I_m}$$

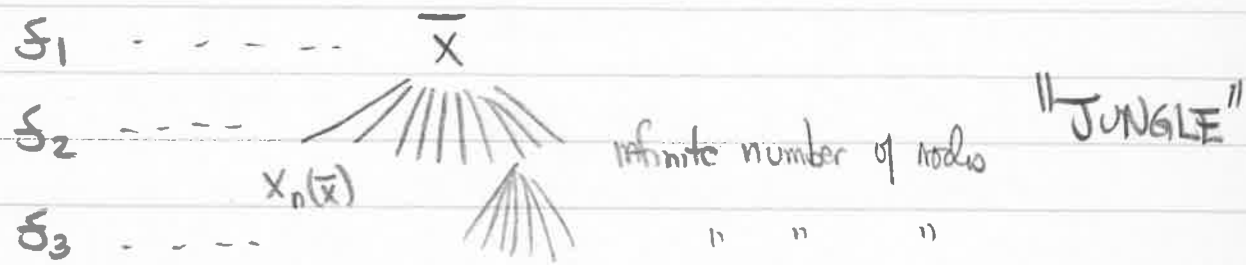
Observe

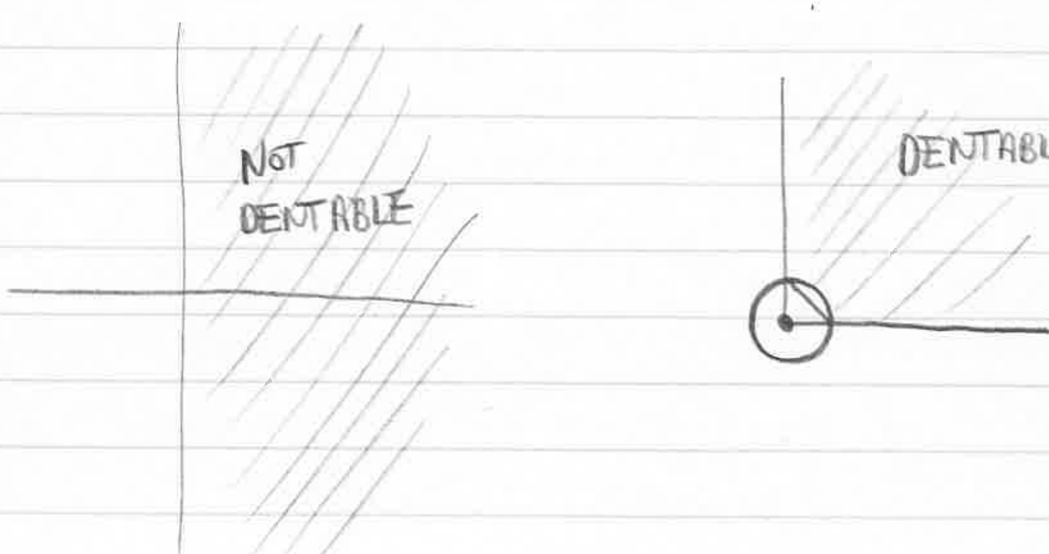
$$\int_{[0,1)} f_2\, d\mu = \sum_{n=1}^{\infty} x_m(\bar{x})\,\alpha_m(\bar{x}) = \bar{x} = \int_{[0,1)} f_1\, d\mu$$

Also observe that

$$\| f_2(t) - f_1(t) \|_X \geq \delta$$

To define f_3, expand each $x_m(\bar{x})$ and chop each I_m appropriately.





THEOREM (Huff, Davis-Phelps)
1973 Duke   1973 PAMS

If X has a bounded non-dentable set D, then there exists a $L_\infty(\mu, X)$ bounded non-L_1-convergent martingale with values in $\overline{co}(D)$.
Consequently bounded not dentable $\Rightarrow$ no RNP

Proof. Let D be the bdd non-dentable set. Chose $\varepsilon>0$ s.t.

$$x \in D \Rightarrow x \in \overline{co}(D \setminus B_\varepsilon(x))$$

We shall produce a sequence (f_n) and a sequence of countable partitions π_n s.t. $\pi_n \le \pi_{n+1}$,

① $$f_n = \sum_{E \in \pi_n} x_E \chi_E \qquad x_E \in D$$

② $$\| f_n(t) - f_{n+1}(t) \|_X \ge \varepsilon \quad \forall n, \forall t$$

③ $$\left\| \int_E (f_m - f_n)\, d\mu \right\| < \frac{\mu(E)}{2^n} \quad \forall E \in \pi_n, \forall m \ge n$$

Given this, we complete the proof as follows. Set

$$F(E) = \lim_n \int_E f_n\, d\mu \quad \forall E \in \cup \pi_n$$

exists by ③

Put

$$g_n = \sum_{E \in \pi_n} \frac{F(E)}{\mu(E)} \chi_E$$

g_n takes values in $\overline{co}(D)$

D bounded implies (g_n) is $L_1(\mu,X)$ bounded and uniformly integrable martingale. Also

$$\int_{[0,1]} \|f_n - g_n\| d\mu = \sum_{E\in\pi_n} \|\chi_E \mu(E) - F(E)\|$$

$$= \lim_m \sum_{E\in\pi_n} \left\| \int_E (f_n - f_m) \, d\mu \right\|$$

$$\leq \sum_{E\in\pi_n} \frac{\mu(E)}{2^n} \longrightarrow 0$$

Obviously (f_n) is not $L_1(\mu,X)$ convergent. Since

$$\int \|f_n - g_n\| d\mu \to 0$$

we see (g_n) is not Cauchy in $L_1(\mu,X)$ norm.

Note: There are two kinds of boundedness for $(f_n) \subset L_1(\mu,X)$

① $\sup_n \|f_n\|_1 < \infty$

② $\sup_n \sup_t \|f_n(t)\|_X < \infty$

Then ② $\Rightarrow$ ① but not conversely. ② also implies uniform integrability

3/13 Vector Measures

Left over: Produce a sequence of countable partitions (π_n) and (f_n) s.t.

(1) $\pi_n \leq \pi_{n+1}$

(2) $f_n = \sum_{E \in \pi_n} x_E \chi_E \qquad x_E \in D$

(3) $\|f_n(t) - f_{n+1}(t)\|_X \geq \varepsilon \quad \forall t \in [0,1)$

(4) $\| \int_E (f_m - f_n)\, d\mu \| < 1/2^n \mu(E) \quad \forall E \in \pi_n$ and $m \geq n$

($\varepsilon > 0$ is such that $x \in D \Rightarrow x \in \overline{co}(D \setminus B_\varepsilon(x))$). Hence $\forall \delta > 0$,

$$x \in D \Rightarrow \|x - \sum_{n=1}^{\infty} \alpha_n(x,\delta)\, x_n(x,\delta)\| < \delta$$

where $\alpha_n(x,\delta) > 0$, $\sum \alpha_n(x,\delta) = 1$, $x_n(x,\delta) \in D \setminus B_\varepsilon(x)$.

Construction: Choose $\bar{x} \in D$ arbitrarily and set

$$f_1 = \bar{x}\, \chi_{[0,1)}$$

$$\pi_1 = \{[0,1)\}$$

disjoint intervals

Suppose π_n and $f_n = \sum_{E \in \pi_n} x_E \chi_E$ have been constructed such that everything is OK

Write for $E \in \pi_n$

$$\|x_E - \sum_{m=1}^{\infty} \alpha_m(x_E, 1/2^{n+1})\, x_m(x_E, 1/2^{n+1})\| < 1/2^{n+1}$$

Put $E=[a,b)$, $\beta_0=0$ $\beta_n=(b-a)\sum_{m=1}^{n}\alpha_m(x_E,\,1/2^{n+1})$. Set

$$I_k=[a+\beta_{k-1},\,a+\beta_k)$$

Define f_{n+1} on E by

$$f_{n+1}\chi_E=\sum_{k=1}^{\infty}x_k(x_E,\,1/2^{n+1})\chi_{I_k}$$

Do this $\forall E\in\pi_n$ and thus define f_{n+1} and π_{n+1} = collection of I_k's
Notice $\pi_n\leq\pi_{n+1}$ and $\|f_{n+1}(t)-f_n(t)\|\geq\varepsilon$ Also $f_{n+1}([0,1))\subset D$
Also if $E\in\pi_n$

$$\left\|\int_E(f_n-f_{n+1})\,d\mu\right\|=\left\|\int_E(x_E\chi_E-f_{n+1})\,d\mu\right\|$$

$$=\left\|x_E-\sum_{m=1}^{\infty}\alpha_m(x_E,\,1/2^{n+1})\,x_m(x_E,\,1/2^{n+1})\right\|\mu(E)$$

$$\leq 1/2^{n+1}\,\mu(E)$$

It follows that $m\geq n$, $E\in\pi_n\Rightarrow$

$$\left\|\int_E(f_m-f_n)\,d\mu\right\|<\frac{1}{2^n}\mu(E)$$

$\boxtimes$

CORPOLLARY: Weakly compact sets are dentable.

(and D not dentable)

Proof. Let D be weakly compact set in last theorem. Let F be as in the last theorem. With the help of chapter 1, extend F to σ-field generated by all π_n's. Observe

$$\left\{ \frac{F(E)}{\mu(E)} : E \in \Sigma \right\} \subseteq \overline{co}(D) \leftarrow \text{weakly compact}$$

Then R-N theorem implies F has a derivative, so (g_n) is a martingale whose limit measure has a derivative $\Rightarrow$ g_n $L_1(\mu, X)$ convergent $\Rightarrow$ f_n is $L_1(\mu, X)$ convergent ↯

▨

non σ-dentable bounded set $\Rightarrow$ no RNP
non dentable bounded set $\Rightarrow$ no RNP

THEOREM: (Rieffel) If every bounded subset of X is σ-dentable, then X has RNP

COROLLARY: X RNP $\Leftrightarrow$ all bounded subsets of X are σ-dentable $\Leftrightarrow$ all bounded subsets of X are dentable.

Proof of Corollary: RNP $\Rightarrow$ dent $\Rightarrow$ σ-dent

(Rieffel's theorem)

Proof of theorem. Let (Ω, Σ, μ) be a finite measure space and X a Banach space such that all bounded subsets of X are σ-dentable. Let $F: \Sigma \to X$ be a μ-continuous vector measure of bounded variation λ. If we can show $dF/d\lambda$ exists, then we'll have

$\uparrow$ μ-cont.

$$\frac{dF}{d\mu} = \frac{dF}{d\lambda}\frac{d\lambda}{d\mu}$$

and we'll be done.

To show: $\forall \varepsilon > 0$ $\forall A \in \Sigma$ with $\lambda(A) > 0$, $\exists B \subset A$ s.t. $\lambda(B) > 0$ and

$$\operatorname{diam}\left\{\frac{F(E)}{\lambda(E)} : E \subset B, \lambda(E) > 0\right\} < 2\varepsilon$$

Let A be a fixed set of positive measure. We know

$$\left\{\frac{F(E)}{\lambda(E)} : E \subset A, \lambda(E) > 0\right\}$$

is bounded, and hence it is σ-dentable. Thus $\exists C \subset A$ with $\lambda(C) > 0$ s.t.

$$\left.\begin{cases} \dfrac{F(C)}{\lambda(C)} = \displaystyle\sum_{n=1}^{\infty} \alpha_n \frac{F(E_n)}{\lambda(E_n)} \\ \alpha_n > 0 \quad \sum \alpha_n = 1 \\ E_n \subset C \end{cases}\right\} \Rightarrow \left\| \frac{F(C)}{\lambda(C)} - \frac{F(E_{n_0})}{\lambda(E_{n_0})} \right\| < \varepsilon$$

for some n_0

If $\sup_{E \subset C} \left\| \frac{F(E)}{\lambda(E)} - \frac{F(C)}{\lambda(C)} \right\| \leq \varepsilon$, then we're done with $B = C$

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$$\lambda(E) = \sup_{\pi} \sum_{A\in\pi} \|F(A)\| \Rightarrow \|F(E)\| \leq \lambda(E)$$

$$\Rightarrow \left\| \frac{F(E)}{\lambda(E)} \right\| \leq 1$$

If not, let j_1 = smallest positive integer ≥ 2 such that $\exists\, C_1 \subset C$ with $\lambda(C_1) > 1/j_1$ and

$$\left\| \frac{F(C_1)}{\lambda(C_1)} - \frac{F(C)}{\lambda(C)} \right\| > \varepsilon$$

Note

$$\frac{F(C)}{\lambda(C)} = \frac{F(C_1)}{\lambda(C_1)} \frac{\lambda(C_1)}{\lambda(C)} + \frac{F(C\setminus C_1)}{\lambda(C\setminus C_1)} \frac{\lambda(C\setminus C_1)}{\lambda(C)}$$

If $\sup\limits_{E \subset C\setminus C_1} \left\| \frac{F(E)}{\lambda(E)} - \frac{F(C)}{\lambda(C)} \right\| \leq \varepsilon$, then we're done with $B = C\setminus C_1$. If not, let j_2 = smallest positive integer ≥ 2 s.t. $\exists\, C_2 \subset C\setminus C_1$ with $\lambda(C_2) > 1/j_2$ and

$$\left\| \frac{F(C_2)}{\lambda(C_2)} - \frac{F(C)}{\lambda(C)} \right\| > \varepsilon$$

Observe

$$\frac{F(C)}{\lambda(C)} = \frac{F(C_1)}{\lambda(C_1)} \frac{\lambda(C_1)}{\lambda(C)} + \frac{F(C_2)}{\lambda(C_2)} \frac{\lambda(C_2)}{\lambda(C)} + \frac{F(C\setminus C_1\setminus C_2)}{\lambda(C\setminus C_1\setminus C_2)} \frac{\lambda(C\setminus C_1\setminus C_2)}{\lambda(C)}$$

Continue. If the process stops, fine, we're done. If not, then continue to produce a disjoint sequence (C_n) of subsets of C s.t. $\lambda(C_n) > 0$

$$\left\| \frac{F(C_n)}{\lambda(C_n)} - \frac{F(C)}{\lambda(C)} \right\| > \varepsilon \quad \forall n$$

[and $E \subset C \setminus C_1 \setminus \cdots \setminus C_{m-1}$ with

$$\left\| \frac{F(E)}{\lambda(E)} - \frac{F(C)}{\lambda(C)} \right\| > \varepsilon \implies \lambda(E) \leq \frac{1}{j_m - 1}$$]

INCORRECT

NOT NEEDED

First observation: $\sum \frac{1}{j_m} \leq \sum \lambda(C_m) < \infty$, and so $j_m \to \infty$

since disjoint

2^{ND} observation

$$\frac{F(C)}{\lambda(C)} = \sum_{n=1}^{\infty} \frac{F(C_n)}{\lambda(C_n)} \frac{\lambda(C_n)}{\lambda(C)} + \frac{F(C \setminus \bigcup_{n=1}^{\infty} C_n)}{\lambda(C \setminus \bigcup_{n=1}^{\infty} C_n)} \frac{\lambda(C \setminus \bigcup_{n=1}^{\infty} C_n)}{\lambda(C)}$$

and so $C \setminus \bigcup_{n=1}^{\infty} C_n$ is not λ-null (otherwise $F(C)/\lambda(C)$ = infinite convex sum of things more than ε away)

Claim: $B = C \setminus \bigcup_{n=1}^{\infty} C_n$ works

Suppose $\exists E \subseteq B = C \setminus C_1 \setminus C_2 \setminus \cdots$ s.t. $\lambda(E) > 0$ and

$$\left\| \frac{F(E)}{\lambda(E)} - \frac{F(C)}{\lambda(C)} \right\| > \varepsilon$$

Then E would have been in the "C_j" hopper for all j. Recall j_m = smallest integer s.t. $\exists C_m \subset C \setminus C_1 \setminus C_2 \cdots \setminus C_{m-1}$ s.t.

$$\lambda(C_m) > 1/j_m \quad \text{and} \quad \left\| \frac{F(C_m)}{\lambda(C_m)} - \frac{F(C)}{\lambda(C)} \right\| \geq \varepsilon$$

III a

$$j_m = \min\left\{k \geq 2 : \exists E \in C \setminus \bigcup_{n=1}^{m-1} C_n \text{ with } \left\| \frac{F(E)}{\lambda(E)} - \frac{F(c)}{\lambda(E)} \right\| > \varepsilon,\ \lambda(E) > 1/k \right\}$$

Since j_{m-1} is not in this set, either

① $j_{m-1} = 1$

or ② $\forall E \in C \setminus \bigcup_{n=1}^{m-1} C_n$ either $\|\cdot\| \leq \varepsilon$ or $\lambda(E) \leq 1/j_{m-1}$

$\exists m_0$ s.t. $j_m > 2$ $\forall m > m_0$. Then if $E \subset C \setminus \bigcup_{n=1}^{m-1} C_n$ and $\|\cdot\| > \varepsilon$, we must have $\lambda(E) \leq 1/j_{m-1}$

But $j_m \to \infty$ and the presence of E forces j_m to stay bounded $\lightning$
In fact if $K =$ smallest integer s.t. $\mu(E) > 1/K$, then all j_m's are $\leq K$.

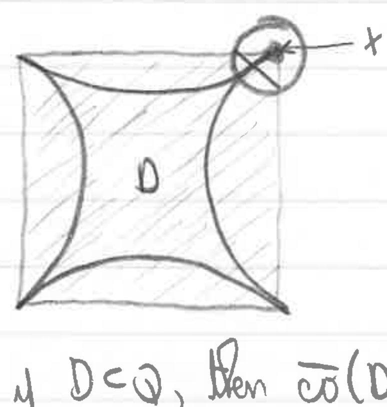
$\boxtimes$

FACTS ABOUT DENTABILITY

① Weakly compact sets are dentable
② A bounded closed convex set is an RNP set $\Leftrightarrow$ each of its subsets is σ-dentable
③ $D \subseteq X$, bounded, $\overline{co}(D)$ dentable $\Rightarrow$ D dentable

Proof. Suppose $\overline{co}(D)$ is dentable and $\varepsilon > 0$. We know $\exists x_\varepsilon \in \overline{co}(D)$ s.t.

$$x_\varepsilon \notin \overline{co}\left(\overline{co}(D) \setminus B_{\varepsilon/2}(x)\right) = Q$$



Notice $D \setminus Q \neq \emptyset$. For if $D \subset Q$, then $\overline{co}(D) \subset Q$, so $x_\varepsilon \in \overline{co}(D) \subset Q$ $\lightning$.

Claim: $d \in D \setminus Q \Rightarrow d \notin \overline{co}(D \setminus B_{\varepsilon/2}(d))$

First note that $d \in B_{\varepsilon/2}(x_\varepsilon)$. For otherwise

$$d \in D \setminus B_{\varepsilon/2}(x_\varepsilon) \subseteq \overline{co}(D \setminus B_{\varepsilon/2}(x_\varepsilon)) \subseteq Q \quad \lightning$$

Hence $D \setminus Q \subseteq B_{\varepsilon/2}(x_\varepsilon)$. It follows that $D \setminus B_\varepsilon(d) \subseteq Q$, for if $d_0 \in D$ and $\|d_0 - d\| \geq \varepsilon$ and $d_0 \notin Q$, then $d, d_0 \in D \setminus Q$, so

$$\|d - d_0\| < \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \lightning$$

Therefore

$$\overline{co}(D \setminus B_\varepsilon(d)) \subset Q$$

whence

$$d \in D \setminus Q \implies d \notin \overline{co}(D \setminus B_\varepsilon(d))$$

$\blacksquare$

(4) If a set D has an exposed point, then D is σ-dentable

Proof. $x \in D$ is exposed by $x^* \in X^*$ if $x^*(x) > x^*(y)$ $\forall y \in D \setminus \{x\}$. So if $x = \sum \alpha_n x_n$, $x_n \in D$ (convex sum), then

$$x^*(x) = \sum \alpha_n x^*(x_n) < \sum \alpha_n x^*(x) = x^*(x)$$

unless all x_n's $= x$

Hence $x \neq \sum \alpha_n x_n$ with $\|x_n - x\| > \varepsilon$.

③ If D has a strongly exposed point, then D is dentable

Proof. $x_0 \in D$ is strongly exposed by $x_0^* \in X^*$ if $x_0^*(x_n) \to x_0^*(x)$ for some sequence (x_n) in D forces $\|x - x_n\| \to 0$ and x is exposed by x_0^*

Suppose $x_0 \in \overline{co}(D \setminus B_\varepsilon(x_0))$. Suppose $\exists \delta > 0$ s.t.

$$x_0^*(x_0) - x_0^*(y) > \delta \quad \forall y \in D \setminus B_\varepsilon(x_0)$$

Then for any convex sum $\sum \alpha_k x_k$ with $x_k \in D \setminus B_\varepsilon(x_0)$,

$$x_0^*(x_0) - x_0^*\left(\sum \alpha_k x_k\right) = \sum \alpha_k \left(x_0^*(x_0) - x_0^*(x_k)\right) > \delta$$

which is impossible since we can get such a convex sum as close to x_0 as desired. Hence there is a sequence $(y_n) \subset D \setminus B_\varepsilon(x_0)$ s.t.

$$x_0^*(y_n) \to x_0^*(x_0)$$

But then $y_n \to x_0$, which is a contradiction since $\|x_0 - y_n\| > \varepsilon$. Hence $x_0 \notin co(D \setminus B_\varepsilon(x_0))$

Note: dentable means $\forall \varepsilon > 0 \; \exists x_\varepsilon \in D$ s.t. $x_\varepsilon \notin \overline{co}(D \setminus B_\varepsilon(x_\varepsilon))$. What we have shown is that if x_0 is a strongly exposed point, then x_0 works for every ε.

3/15 VECTOR MEASURES

$D \subset X$ (bounded) is an RNP set if $\forall (\Omega, \Sigma, \mu)$ $\forall$ c.a. $F: \Sigma \to X$ st.

$$\frac{F(E)}{\mu(E)} \in D \quad \forall E \text{ with } \mu(E) \neq 0$$

then $F = \int f \, d\mu$.

Bourgain: A closed convex bounded D is a RNP set iff each of its convex closed bounded subsets is the closed convex hull of its strongly exposed points.

THEOREM: (Bishop-Phelps) Let C be a closed bounded convex subset of a Banach space. Then the set of x^* in X^* that attain their supremum on C is norm dense in X^*.

Proof in book.

THEOREM (Lindenstrauss) X RNP $\overset{①}{\Longrightarrow}$ every closed bounded convex subset of X has an extreme point $\overset{②}{\Longrightarrow}$ every closed bounded convex subset of X is the norm closed convex hull of its extreme points.

(Remark - Last condition is the Krein-Milman property KMP. Huff & Morris showed that KMP $\Rightarrow$ RNP for dual spaces. Open in general

Proof. Assume ① has been done. To prove ② let B be a closed convex bounded subset of X. Let

$$E = \overline{co}\,(\text{ext } B)$$

Obviously $E \subset B$. If $x \in B \setminus E$, then by Hahn-Banach (separation form), there exists $x^* \in X^*$ s.t.

$$x^*(x) > \alpha > \sup x^*(E)$$

By Bishop-Phelps we can find $y^* \in X^*$ s.t. y^* achieves its max on B and

$$\sup y^*(B) > \alpha > \sup y^*(E)$$

Select $b \in B$ such that $y^*(b) = \sup y^*(B)$. Put

$$C := \{ z \in B : y^*(z) = y^*(b) \}$$

Then C is a closed bounded convex subset of B. Hence C has an extreme point $c \notin E$. We'll be done if we show c is an extreme point of B. Suppose

$$c = t b_1 + (1-t) b_2$$

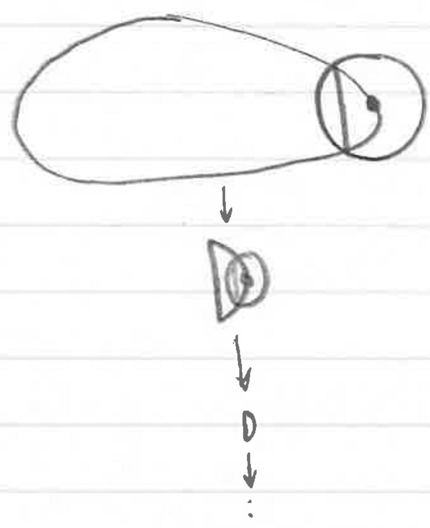
for $0 < t < 1$, $b_1, b_2 \in B$. Then

$$y^*(c) = t\, y^*(b_1) + (1-t)\, y^*(b_2) = y^*(b) = \sup y^*(B)$$

$$\Rightarrow y^*(b_1) = \underset{= y^*(b)}{y^*(b_2)} \Rightarrow b_1, b_2 \in C$$

Hence $b_1 = b_2$ since $c \in \mathrm{ext}(C)$

Proof of ①: Intuition for ①.



(This proof works for D s.t. all subsets of D are dentable)

Let $D \subset X$ be closed bounded convex. Since X has RNP, D is dentable, so $\exists\, x_1 \notin \overline{co}(D \setminus B_{1/2}(x_1)) =: C_1$. Choose $x^* \in X^*$ s.t.

$$x^*(x_1) > \alpha > \beta > \sup x^*(C_1)$$

By Bishop-Phelps there exist a max attaining x_1^* and $z_0 \in D \setminus C_1$ s.t.

$$\sup x_1^*(D) = x_1^*(z_0) > \alpha > \beta > \sup x_1^*(C_1)$$

Put $D_1 := \{ x \in D : x_1^*(x) = x_1^*(z_0) \}$. Then $\mathrm{diam}(D_1) < 2 \cdot 1/2 = 1$

since $D_1 < B_{1/2}(x_1)$.

Notice D_1 is a "face" of D, i.e. closed convex bounded.
Also D_1 is dentable (since X has RNP). Hence $\exists\, x_2 \in D_1$ s.t.

$$x_2 \notin \overline{co}\left(D_1 \setminus B_{1/4}(x_2)\right)$$

Apply Bishop-Phelps to find x_2^* and $z_1 \in D_1 \setminus C_2$ st.

$$\sup x_2^*(D_1) = x_2^*(z_1) > \sup x_2^*(C_2)$$

Put $D_2 = \{ z \in D_1 : x_2^*(z) = x_2^*(z_1)\}$. Then $\operatorname{diam}(D_2) < 1/2^2$.
Continue this to get

$$D \supset D_1 \supset D_2 \supset \dots$$

such that $D_n \neq \emptyset$ and $\operatorname{diam}(D_n) \to 0$.

$$D_{n+1} = \{ z \in D_n : x_{n+1}^*(z) = \sup x_{n+1}^*(D_n)\}$$

By completeness, $\bigcap_{n=1}^{\infty} D_n = \{x\}$ for some x. Claim: x is an extreme point of D. If

$$x = t y_1 + (1-t) y_2$$

for some y_1 and y_2 in D, then

$$\forall n \qquad x_{n+1}^*(x) = t x_{n+1}^*(y_1) + (1-t) x_{n+1}^*(y_2)$$

Hence definition of D_n forces y_1, y_2 to be in all D_n's. Therefore

$$y_1, y_2 \in \bigcap D_n = \{x\}$$

whence $y_1 = y_2 = x$.

$\blacksquare$

RNP for $L_p(\mu, X)$

(Ω, Σ, μ)

<u>**THEOREM**</u>: The space $L_p(\mu, X)$ has RNP iff $1 < p < \infty$ and X has RNP.

Proof ($\Rightarrow$) Trivial since $\{x \chi_\Omega : x \in X\}$ is a copy of X in $L_p(\mu, X)$. Also, if we take $\bar{x} \in X$ with $\|\bar{x}\| = 1$, then

$$\{\bar{x}\varphi : \varphi \in L_p(\mu)\}$$

is an isometric copy of $L_p(\mu)$ in $L_p(\mu, X) \Rightarrow p \neq 1$

($\Leftarrow$) Let $(S, \mathcal{F}, \lambda)$ be a finite measure space and $F : \mathcal{F} \to L_p(\mu, X)$ be a λ-continuous vector measure of bounded variation. WLOG

$$\|F(E)\|_{L_p(\mu, X)} \leq \lambda(E)$$

$\forall E \in \mathcal{F}$. If $w \in \Omega$, $s \in S$ and a Σ-partition π and $\mathcal{F}$-partition Δ

Then write

$$f_{\pi,\Delta}(\omega,s) = \sum_{I\in\pi}\sum_{E\in\Delta} \frac{\int_I F(E)\,d\mu}{\lambda(E)\mu(I)} \chi_E(s)\chi_I(\omega)$$

This defines a martingale in $L_p(\lambda\times\mu, X)$. We'll prove this martingale is bounded and hence $L_p(\lambda\times\mu, X)$ convergent. To this end, notice

$$\left\| \int_I F(E)\,d\mu \right\|_X^p = \left\| \int_I F(E)\,\chi_I\,d\mu \right\|_X^p$$

$$\overset{\text{Holder}}{\leq} \| F(E)\chi_I \|_{L_p}^p \; \mu(I)^{p/q}$$

Hence

$$\| f_{\pi,\Delta} \|_{L_p(\lambda\times\mu, X)}^p = \sum_{I\in\pi}\sum_{E\in\Delta} \frac{\left\| \int_I F(E)\,d\mu \right\|_X^p}{\lambda(E)^p \mu(I)^p} \mu(I)\lambda(E)$$

$$\leq \sum_{I\in\pi}\sum_{E\in\Delta} \frac{\| F(E)\chi_I \|_{L_p(\mu,X)}^p \, \mu(I)^{\overbrace{1+p/q-p}^{=0}} \lambda(E)}{\lambda(E)^p}$$

additive function of I

$$= \sum_{E\in\Delta} \frac{\| F(E) \|_{L_p(\mu,X)}^p}{\lambda(E)^p} \lambda(E)$$

$$\leq \sum_{E\in\Delta} \lambda(E) = \lambda(S) < \infty$$

Hence $(f_{\pi,\Delta})$ is $L_p(\lambda \times \mu, X)$ bounded and hence convergent. Let its $L_p(\lambda\times\mu, X)$ limit be f. Then

$$\int_{\Omega\times S} \|f(\omega,s)\|^p \, d\mu(\omega)\, d\lambda(s) < \infty$$

Therefore $f(\cdot, s) \in L_p(\mu, X)$ for λ-almost all s

Redefine f to be zero on the exceptional set and ignore any measurability problems. Put

$$g(s) := f(\cdot, s) \quad \forall s \in S$$

It is easy to show that the $L_p(\mu, X)$ valued function g is measurable (D-S III.17) and Bochner integrable. If $A \in \mathcal{F}$, then

$$\int_A g\, d\lambda = \lim_{\pi,\Delta} \int_A \sum_{I\in\pi} \sum_{E\in\Delta} \frac{\int_I F(E)\, d\mu}{\mu(I)\lambda(E)} \chi_I \chi_E \, d\lambda$$

$$= \lim_{\Delta} \int_A \sum_{E\in\Delta} \underbrace{\lim_{\pi} \Big(\sum_{I\in\pi} \frac{\int_I F(E)\, d\mu\, \chi_I}{\mu(I)} \Big)}_{\text{martingale} \to F(E)} \frac{\chi_E}{\lambda(E)} \, d\lambda$$

$$= \lim_{\Delta} \int_A \sum_{E\in\Delta} \frac{F(E)}{\lambda(E)} \chi_E \, d\lambda = F(A)$$

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Let Δ be a refinement of $\{S \setminus A, A\}$. Then $A = \bigcup_{\substack{E \in \Delta \\ E \subset A}}$ and so

$$\int_A \sum_{E \in \Delta} \frac{F(E)}{\lambda(E)} \chi_E \, d\lambda = \sum_{\substack{E \in \Delta \\ E \subset A}} \frac{F(E)}{\lambda(E)} \lambda(E) = \sum_{\substack{E \in \Delta \\ E \subset A}} F(E) = F(A)$$

3/27 VECTOR MEASURES

THEOREM: Let $1 \leq p < \infty$. Then $(L_p(\mu, X))^* = L_q(\mu, X^*)$ if and only if X^* has RNP w.r.t. μ

Proof. Let $g \in L_q(\mu, X^*)$. Note $\int_\Omega \langle f, g \rangle \, d\mu$ exists $\forall f \in L_p(\mu, X)$ by Holder, and

$$\left| \int_\Omega \langle f, g \rangle \, d\mu \right| \leq \|f\|_p \|g\|_q$$

Hence

$$l_g(\cdot) = \int \langle \cdot, g \rangle \, d\mu$$

is a bounded linear functional on $L_p(\mu, X)$. Also $\|l_g\| \leq \|g\|_q$
Want to show $\|l_g\| = \|g\|_q$.

First suppose

$$g = \sum_{i=1}^{\infty} x_i^* \chi_{E_i}$$

disjoint positive measure

Let $\varepsilon > 0$ and choose $h \geq 0$ in $L_p(\mu)$ s.t. $0 < \|h\|_p \leq 1$ and

$$\|g\|_q - \varepsilon/2 < \int_\Omega \|g\| h \, d\mu$$

Choose $x_i \in X$ s.t. $\|x_i\| = 1$ and

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Let $g \in L_q(\mu, X^*)$. $\langle f, g\rangle(\omega) := g(\omega)(f(\omega))$. Let g_n be simple functions converging a.e. to g.

$$\langle f, g_n\rangle(\omega) = \langle f, \sum_{i=1}^{n} x_i^* \chi_{E_i}\rangle(\omega) = \sum_{i=1}^{n} \underbrace{x_i^*(f(\omega))}_{\text{measurable}} \underbrace{\chi_{E_i}(\omega)}_{\text{measurable}}$$

Hence $\langle f, g_n\rangle$ is measurable. Also $g_n(\omega) \to g(\omega) \Rightarrow g_n(\omega)(f(\omega)) \to g(\omega)(f(\omega))$
$\Rightarrow \langle f, g_n\rangle \longrightarrow \langle f, g\rangle$ a.e. Therefore $\langle f, g\rangle$ is measurable

$\|g(\cdot)\|_X \in L_q(\mu)$

$$\|x_i^*\| - \frac{\varepsilon}{2\|h\|_1} < x_i^*(x_i)$$

Set

$$f := \sum_{i=1}^{\infty} x_i h \chi_{E_i}$$

Then $\|f\|_p = \|h\|_p \leq 1$, and

$$l_g(f) = \int_\Omega \langle f, g\rangle d\mu = \int h \sum_{i=1}^{\infty} x_i^*(x_i)\chi_{E_i}\, d\mu$$

$$\geq \int h \sum_{i=1}^{\infty} \left(\|x_i^*\| - \frac{\varepsilon}{2\|h\|_1}\right)\chi_{E_i}\, d\mu$$

$$\geq \int_\Omega h\|g\|\, d\mu - \varepsilon/2$$

$$\geq \|g\|_q - \varepsilon/2 - \varepsilon/2$$

Hence $\|l_g\| \geq \|g\|_q$. Holds for general g since countably valued functions dense.

This proves that $L_q(\mu, X^*)$ is isometric to a subspace of $L_p(\mu, X)^*$ (no assumption of X RNP is needed here)

Now assume X^* has RNP w.r.t. μ. Let $l \in L_p(\mu, X)^*$.

Define

$$G(E)(x) = l(x\chi_E) \quad x \in X,\ E \in \Sigma$$

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Let $g \in L_q(\mu, X^*)$ and choose a seq. (g_n) of countably valued functions in $L_q(\mu, X^*)$ s.t. $\lim \|g_n - g\|_q = 0$. Let

$$\ell_n(\cdot) = \int \langle \cdot, g_n \rangle \, d\mu \qquad \ell(\cdot) = \int \langle \cdot, g \rangle \, d\mu$$

Then by what we've shown $\|\ell_n\| = \|g_n\|_q$, and $0 \leq \|\ell_n - \ell\| \leq \|g_n - g\| \to 0$
Hence $\|\ell\| = \lim \|\ell_n\| = \lim \|g_n\|_q = \|g\|_q$

Observe that $\|G(E)x\| \leq \|\ell\| \, \|x\chi_E\|_p = \|\ell\| \, \|x\| \, \|\chi_E\|_p$.
Therefore $G : \Sigma \to X^*$ and $G \ll \mu$ since

$$\|\chi_E\|_p \to 0 \quad \text{as} \quad \mu(E) \to 0$$

Claim: G is of bounded variation. To see this consider

$$\Big| \sum_{E \in \pi} G(E)(x_E) \Big|$$

where $\|x_E\| = 1$.

$$\begin{aligned} \Big| \sum_{E \in \pi} G(E) x_E \Big| &= \Big| \ell\Big(\sum_{E \in \pi} x_E \chi_E \Big) \Big| \\ &\leq \|\ell\| \, \Big\| \sum_{E \in \pi} x_E \chi_E \Big\|_p \\ &\leq \|\ell\| \, \Big\| \sum_{E \in \pi} \chi_E \Big\|_p \\ &= \|\ell\| \, \mu(\Omega)^{1/p} \end{aligned}$$

By choosing x_E properly we conclude that G is of bounded variation.
Hence by RNP for X^* there exists a Bochner integrable $g : \Omega \to X^*$ s.t.

$$g(E) = \int_E g \, d\mu$$

If we can show $g \in L_q(\mu, X^*)$ and $\ell(f) = \int \langle f, g \rangle \, d\mu$, then we'll have shown previous isometry is onto.

To this end, observe that

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$$\|G(E)\| \leq \|\ell\| \, \|\chi_E\|_p = \|\ell\| \, \mu(E)^{1/p}$$

For each $E \in \pi$ choose x_E s.t. $\|G(E)\| - \varepsilon/n \leq G(E)x_E$ (where $n = \#\pi$), $\|x_E\| = 1$

Then

$$\sum_{E\in\pi} \|G(E)\| - \varepsilon \leq \sum_{E\in\pi} G(E)x_E \leq \left| \sum_{E\in\pi} G(E)x_E \right| \leq \|\ell\| \, \mu(\Omega)^{1/p}$$

Hence $\sup_{\pi} \sum_{E\in\pi} \|G(E)\| \leq \|\ell\| \, \mu(\Omega)^{1/p} < \infty$

$$\ell(f) = \int_\Omega \langle f, g\rangle d\mu$$

for all simple $f \in L_p(\mu, X)$. Choose $E_n \uparrow \Omega$ s.t. $g\chi_{E_n}$ is bounded. Then $g\chi_{E_n} \in L_q(\mu, X^*)$ for all n. Therefore

$$\int_{E_n} \langle \cdot, g\rangle d\mu \in L_p(\mu, X^*)$$

In fact

$$\ell(f\chi_{E_n}) = \int_{E_n} \langle f, g\rangle d\mu \quad \forall f \in L_p(\mu, X)$$

Furthermore

$$\|\ell\| \geq \|\ell(\cdot\chi_{E_n})\| = \|g\chi_{E_n}\|_q \quad \forall n \in \mathbb{N}$$

by previous work

i.e.

$$\int_\Omega \|g\|^q \chi_{E_n} d\mu \leq \|\ell\|^q$$

By monotone convergence $\|g\|^q \in L_1(\mu, X^*) \Rightarrow g \in L_q(\mu, X^*)$. Now it is easy to see that $\ell(f) = \int \langle f, g\rangle d\mu$ for all $f \in L_p(\mu, X)$.

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If $f = \sum x_i \chi_{E_i}$, then

$$\ell(f) = \sum \ell(x_i \chi_{E_i}) = \sum G(E_i) x_i = \sum \left(\int_{E_i} g \, d\mu \right) x_i$$

$$= \int \sum x_i \chi_{E_i} g \, d\mu = \int \langle f, g \rangle \, d\mu$$

To prove the converse, let $G : \Sigma \to X^*$ be a μ-continuous vector measure of bounded variation. The measure $|G|$ is also μ-continuous and so if $A \in \Sigma$ has positive measure, $\exists\ B \subset A$, $\mu(B) > 0$ and a $\beta > 0$ s.t.

$$|G|(E \cap B) \leq \beta\, \mu(E \cap B)$$

$\ulcorner$ To get β, B

$$|G|(E \cap A) = \int_{E \cap A} \varphi\, d\mu \quad \forall E \in \Sigma$$

Let β be such that

$$B := A \cap [\varphi \leq \beta]$$

has positive μ-measure. $\lrcorner$

Claim: There exists a Bochner integrable g supported on B s.t.

$$G(E \cap B) = \int_{E \cap B} g\, d\mu$$

By hypothesis $L_p(\mu, X)^* = L_q(\mu, X^*)$. Define ℓ on $L_p(\mu, X)$ by

$$\ell\left(\sum_{i=1}^{n} x_i \chi_{E_i}\right) := \sum_{i=1}^{n} G(E_i \cap B) x_i$$

Observe that

$$\left| \ell\left(\sum x_i \chi_{E_i}\right) \right| = \left| \sum_{i=1}^{n} \frac{G(E_i \cap B)}{\mu(E_i \cap B)} \mu(E_i \cap B) x_i \right|$$

$$\leq \sum_{i=1}^{n} \left\| \frac{G(E_i \cap B)}{\mu(E_i \cap B)} \right\| \|x_i\| \mu(E_i \cap B)$$

$$\leq \beta \left\| \sum x_i \chi_{E_i \cap B} \right\|_{L_1}$$

$$\leq \beta \left\| \sum x_i \chi_{E_i} \right\|_1$$

$$\leq \beta \left\| \sum x_i \chi_{E_i} \right\|_p \mu(\Omega)^{1/q}$$

Therefore ℓ extends to a member of $L_p(\mu, X)^*$. Hence $\exists\, g \in L_q(\mu, X^*)$ such that

$$\ell(f) = \int_\Omega \langle f, g \rangle \, d\mu \quad \forall f \in L_p(\mu, X)$$

Moreover, if $E \in \Sigma$ and $x \in X$, then

$$G(E \cap B)\, x = \ell(x \chi_{E \cap B}) = \int_\Omega \langle x \chi_{E \cap B}, g \rangle \, d\mu$$

$$= \left(\int_{E \cap B} g \, d\mu \right) x$$

Therefore

$$G(E\cap B)=\int_{E\cap B} g\,d\mu \quad \forall E$$

$\blacksquare$

<u>COROLLARY</u>: $L_p(\mu, X)$ is reflexive iff X is reflexive and $1<p<\infty$

<u>COROLLARY</u>: Hilbert spaces have RNP

Proof. $L_2(\mu,H)$ is a Hilbert space, so

$$L_2(\mu,H)^* = L_2(\mu,H) = L_2(\mu,H^*)$$

Hence $H=H^*$ has RNP.

General Vector Measure Theory

$\mathcal{F}$ will denote a field of subsets of Ω
Σ will denote a σ-field of subsets of Ω

__Definition:__ A (finitely additive) vector measure $F : \mathcal{F} \to X$ is countably additive if

$$F\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} F(E_n)$$

for all disjoint sequences $(E_n) \subset \mathcal{F}$ with $\bigcup E_n \in \mathcal{F}$.

__Definition:__ A finitely additive $F : \mathcal{F} \to X$ is strongly additive if $\sum F(E_n)$ exists for all disjoint sequences $(E_n) \subset \mathcal{F}$.

__Definition:__ The semi-variation of F is

$$\|F\|(E) := \sup_{\|x^*\| \le 1} |x^* F|(E)$$

__Examples__

1 i) $F : \underset{\substack{\uparrow \\ \text{Borel sets in } [0,1]}}{\Sigma} \longrightarrow L_p(\Sigma, \underset{\substack{\uparrow \\ \text{Lebesgue measure on } [0,1]}}{\mu})$ given by $F(E) = \chi_E$

Then if

ⓐ $p=\infty$ F is not countably or strongly additive

E_n disjoint seq. of sets of positive measure. Then $\sum F(E_n)$ is not convergent since $\|F(E_n)\|=1 \ \forall n$

ⓑ $1 \le p < \infty$ F is countably additive (by dominated convergence)

ⓒ $p=1$ F is of bounded variation

$$\sum_{E\in\pi} \|F(E)\|_1 = \sum_{E\in\pi} \mu(E) = \mu[0,1]$$

ⓓ $1<p<\infty$ $|F|$ is purely 0-∞ valued (i.e. takes only the values 0 or $+\infty$)

If E is a μ-null set, then $|F|(E)=0$. If $\mu(E)>0$ then divide E into n parts $\underset{A_1,\dots,A_n}{\wedge}$ each with measure $\mu(E)/n$.

$$\sum_{i=1}^{n} \|F(A_i)\|_p = \sum_{i=1}^{n} \mu(A_i)^{1/p} = \sum_{i=1}^{n} \frac{\mu(E)^{1/p}}{n^{1/p}}$$

$$= \mu(E)^{1/p}\, n^{1-1/p} = \mu(E)^{1/p}\, n^{1/q} \to \infty \text{ as } n\to\infty$$

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$$\| F(\bigcup_{n=1}^{\infty} E_n) - \sum_{n=1}^{k} F(E_n) \|_p^p = \int | \chi_{\bigcup_{n=1}^{\infty} E_n} - \sum_{n=1}^{k} \chi_{E_n} |^p \, d\mu$$

Now $| \chi_{\bigcup_{n=1}^{\infty} E_n} - \sum_{n=1}^{k} \chi_{E_n} | \longrightarrow 0$ pointwise and is dominated by constant function 2.

By D.C.T. $\| F(\bigcup_{n=1}^{\infty} E_n) - \sum_{n=1}^{k} F(E_n) \| \longrightarrow 0$

FACT: F bounded variation $\Rightarrow |F|$ is finitely additive

"Proof"

$$|F|(A) = \lim_{\pi_A} \sum_{E \in \pi_A} \|F(E)\|$$

<u>THEOREM</u>: Let $F : \mathcal{F} \to X$ be finitely additive and of bounded variation. Then $|F|$ is countably additive iff F is countably additive

<u>THEOREM</u>: Let μ be a finite non-negative countably additive measure on $\mathcal{F}$. If $F : \mathcal{F} \to X$ is a finitely additive vector measure and $F \ll \mu$, then F is countably additive

Proof. Let $(E_n) \subset \mathcal{F}$ be disjoint with $\bigcup E_n \in \mathcal{F}$.

$$F\left(\bigcup_{n=1}^{\infty} E_n\right) - \sum_{n=1}^{k} F(E_n) = F\left(\bigcup_{n=1}^{\infty} E_n\right) - F\left(\bigcup_{n=1}^{k} E_n\right)$$

$$= F\left(\bigcup_{n=k+1}^{\infty} E_n\right) \to 0 \quad \text{by } \mu\left(\bigcup_{n=k+1}^{\infty} E_n\right) \to 0$$

Hence $F\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} F(E_n)$

▨

Proof of penultimate theorem.

Proof. Since $F \ll |F|$, last theorem shows $|F|$ c.a. $\Rightarrow F$ c.a.

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A,B disjoint

① $$\sum_{E\in\Pi_{A\cup B}} \|F(E)\| = \sum_{E\in\Pi_{A\cup B}} \|F(E\cap A)+F(E\cap B)\|$$

$$\leq \sum_{E\in\Pi_{A\cup B}} \|F(E\cap A)\| + \sum_{E\in\Pi_{A\cup B}} \|F(E\cap B)\|$$

$$\leq |F|(A)+|F|(B)$$

Hence $|F|(A\cup B) \leq |F|(A)+|F|(B)$

② Let $\varepsilon>0$. Let Π_A, Π_B be partitions of A,B respectively such that

$$|F|(A) \leq \sum_{E\in\Pi_A} \|F(E)\| + \varepsilon/2$$

$$|F|(B) \leq \sum_{E\in\Pi_B} \|F(E)\| + \varepsilon/2$$

Then $\Pi_A\cup\Pi_B$ is a partition of $A\cup B$ (since A,B disjoint) and

$$|F|(A)+|F|(B) \leq \sum_{E\in\Pi_A\cup\Pi_B} \|F(E)\| + \varepsilon \leq |F|(A\cup B)+\varepsilon$$

Since ε is arbitrary, $|F|(A)+|F|(B) \leq |F|(A\cup B)$

Let (E_n) be a disjoint seq. in $\mathcal{F}$ s.t. $B = \bigcup_{n=1}^{\infty} E_n \in \mathcal{F}$. Let π be any partition of B

$$\sum_{A\in\pi} \|F(A)\| = \sum_{A\in\pi} \|F(A \cap \bigcup_{n=1}^{\infty} E_n)\|$$

$$= \sum_{A\in\pi} \|\sum_{n=1}^{\infty} F(A\cap E_n)\|$$

F c.a.

$$\leq \sum_{A\in\pi}\sum_{n=1}^{\infty} \|F(A\cap E_n)\|$$

$$= \sum_{n=1}^{\infty}\sum_{A\in\pi} \|F(A\cap E_n)\|$$

$$\leq \sum_{n=1}^{\infty} |F|(E_n)$$

Hence

$$|F|\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} |F|(E_n)$$

On the otherhand,

$$\sum_{n=1}^{k} |F|(E_n) = |F|\left(\bigcup_{n=1}^{k} E_n\right) \leq |F|\left(\bigcup_{n=1}^{\infty} E_n\right)$$

and so

$$\sum_{n=1}^{\infty} |F|(E_n) \leq |F|\left(\bigcup_{n=1}^{\infty} E_n\right) \qquad \blacksquare$$

3/29 VECTOR MEASURE

Facts

① $$\|F\|(A) = \sup_{\pi_A}\left\{\left\| \sum_{E\in\pi} \varepsilon_E F(E) \right\| : |\varepsilon_E| \leq 1 \right\}$$

② $$\sup_{E\subset A} \|F(E)\| \leq \|F\|(A) \leq 4 \sup_{E\subset A} \|F(E)\|$$

(i.e. bounded semi-variation $\Leftrightarrow$ bounded range)

Proof. (1)

$$\left\| \sum_{E\in\pi_A} \varepsilon_E F(E) \right\| = \sup_{\|x^*\|\leq 1} \left| \sum_{E\in\pi_A} x^* \varepsilon_E F(E) \right| \leq \sup_{\|x^*\|\leq 1} \sum_{E\in\pi_A} |x^* F(E)|$$

$$\leq \|F\|(A)$$

Also if $\|x^*\| \leq 1$ and π is a partition of A, then

$$\sum_{E\in\pi} |x^* F(E)| = x^* \sum_{E\in\pi} \operatorname{sgn}(x^* F(E))\, F(E) \qquad \text{(Real case)}$$

$$\leq \left\| \sum_{E\in\pi} \operatorname{sgn}(x^* F(E))\, F(E) \right\|$$

Now take sups.

(2) We have

$$\sup_{E\subset A} \|F(E)\| = \sup_{E\subset A} \sup_{\|x^*\|\le 1} |x^* F(E)|$$

$$\le \sup_{E\subset A} \sup_{\|x^*\|\le 1} |x^* F|(E)$$

$$= \sup_{\|x^*\|\le 1} |x^* F|(A) = \|F\|(A)$$

For the other inequality, consider for a partition π and $\|x^*\| \le 1$

$$\sum_{E\in\pi} |x^* F(E)| = \sum_{E\in\pi^+} x^* F(E) - \sum_{E\in\pi^-} x^* F(E)$$

$$\le \Big|x^* F\Big(\bigcup_{E\in\pi^+} E\Big)\Big| + \Big|x^* F\Big(\bigcup_{E\in\pi^-} E\Big)\Big|$$

Then

$$\sup_{\|x^*\|\le 1} \sum_{E\in\pi} |x^* F(E)| \le 2 \sup_{E\subset A} \|F(E)\|$$

Now take sup over π. (In complex case would get 4 instead of 2)

$\boxtimes$

Let $T: B(\mathcal{F}) \longrightarrow X$, where $B(\mathcal{F})$ is all uniform limits of $\mathcal{F}$-simple functions in sup norm. Then

$$\|T\| = \sup_{\|f\| \leq 1} \|T(f)\| = \sup_{\|\sum_{A \in \pi} \varepsilon_A \chi_A\| \leq 1} \|\sum \varepsilon_A T(\chi_A)\|$$

$$= \sup_{\pi} \sup_{|\varepsilon_A| \leq 1} \|\sum_{A \in \pi} \varepsilon_A T(\chi_A)\|$$

$$= \|F\|(\Omega) \qquad \text{where } F(E) := T(\chi_E)$$

(Finitely additive)

Read this backwards as follows: Let $F: \mathcal{F} \longrightarrow X$ be a vector measure with bounded range. Define T: $\mathcal{F}$-simple functions to X by

$$T\left(\sum \alpha_i \chi_{E_i}\right) := \sum \alpha_i F(E_i)$$

Then T is bounded with norm $\|F\|(\Omega)$. Hence $B(B(\mathcal{F}), X)$ is the set of all vector measures $F: \mathcal{F} \to X$ with bounded range, where

$$T(f) = \int_\Omega f \, dF$$

$$\|T\| = \|F\|(\Omega)$$

[$B(L_\infty(\mu), X)$ = same thing provided you demand F vanish on μ-null sets]

Note "s.a" = "s.bdd" = "exhaustive"

PROPOSITION: $F : \mathcal{F} \to X$ is strongly additive iff $\lim F(E_n) = 0$ for all disjoint sequences (E_n) in $\mathcal{F}$. Consequently if there exists finitely additive non-negative μ on $\mathcal{F}$ s.t. $F \ll \mu$, then F is strongly additive.

Proof. If F is strongly additive, clearly $\lim F(E_n) = 0$. If F is not strongly additive, then there exist disjoint sequence (E_n) s.t. partial sums of $\sum F(E_n)$ are not Cauchy, i.e. there exist blocks of positive $A_1, A_2, \ldots$ s.t. $\max A_i < \min A_{i+1}$ and
finite

$$\Big\| \sum_{j \in A_i} F(E_j) \Big\| \geq \varepsilon \quad \forall i$$

Set

$$B_i = \bigcup_{j \in A_i} E_j$$

Then $\|F(B_i)\| \geq \varepsilon$ $\forall i$ and (B_i) is a disjoint sequence.

$\boxtimes$

THEOREM: Any of the following statements about a family $\{F_\tau : \tau \in T\}$ of finitely additive vector measures of $\mathcal{F}$ implies all the others:

(1) $\{F_\tau\}$ is uniformly s.a.

(2) $\forall$ disjoint (E_n) in $\mathcal{F}$, $\|F_\tau(E_n)\| \to 0$ uniformly in τ

(3) $\forall$ disjoint (E_n) in $\mathcal{F}$, $\|F_\tau\|(E_n) \to 0$ uniformly in τ

(4) $\{ |x^* F_\tau| : \|x^*\| \leq 1, \tau \in T \}$ is uniformly s.a.

<u>THEOREM</u>: TFAE for a finitely additive vector measure $F : \mathcal{F} \longrightarrow X$

(1) F is strongly additive

(2) $\{x^*F : \|x^*\| \leq 1\}$ is unif. s.a.

(3) $F(E_n) \to 0 \quad \forall$ disjoint (E_n)

(4) $\|F\|(E_n) \to 0 \quad \forall$ disjoint (E_n)

(5) $\{|x^*F| : \|x^*\| \leq 1\}$ is unif. s.a.

(6) $\lim_n F(E_n)$ exists $\forall$ monotone sequences (E_n)

FACT – s.a. $\Rightarrow$ bdd range

<u>THEOREM</u> (Pettis) Let $F : \underset{\sigma\text{-field}}{\Sigma} \longrightarrow X$ be countably additive. Let μ be countably additive non-negative (finite) measure on Σ. Then $F \ll \mu$ if and only if F vanishes on μ-null sets

Proof. $\Rightarrow$ obvious

$(\Leftarrow)$ Suppose F is not such that $F \ll \mu$. Then $\exists\, \varepsilon > 0$ s.t. $\forall n$ there exists A_n with

$$\|F(A_n)\| \geq \varepsilon \qquad \mu(A_n) < 1/2^n$$

Take $x_n^* \in X^*$ with $\|x_n^*\| = 1$ such that

$$x_n^* F(A_n) = \|F(A_n)\| \geq \varepsilon$$

The family $\{|x_n^* F| : n \in \mathbb{N}\}$ is unif. s.a. Set $B_n = \bigcup_{j=n}^{\infty} A_j$

Notice $B_n \downarrow B$, say, and $\mu(B) = 0$. Hence F vanishes on all subsets of B, so

$$|x_n^* F|(B) = 0 \quad \forall n$$

Then $|x_j^* F|(B_n) \longrightarrow 0$ uniformly in j because $|x_j^* F|$ is unif. c.a.
But

$$\varepsilon \leq \|x_n^* F(A_n)\| \leq |x_n^* F|(A_n) \leq |x_n^* F|(B_n) \quad \lightning$$

$\boxtimes$

This theorem if false for c.a. measures on fields even in the scaler case.

Example: Let $\lambda \in L_\infty^*(\mu) \setminus L_1(\mu)$. λ is a finitely additive measure on Σ that vanishes on μ-null sets. By Stone's representation theorem there exists a totally disconnected compact Hausdorff space V and a Boolean isomorphism $\tau : \Sigma \to$ (the field of all clopen subsets of V) $= \Sigma^*$. Define $\bar{\mu}$ on Σ^* by

$$\bar{\mu}(\tau(E)) = \mu(E)$$

Also

$$\bar{\lambda}(\tau(E)) = \lambda(E)$$

Observe that $\bar{\lambda}$ vanishes on $\bar{\mu}$-null sets. Also observe that if (S_n) is a disjoint seq. in Σ^* with $\cup S_n \in \Sigma^*$, then (S_n) is an open cover of $\cup S_n$ = compact set. Therefore all but finitely many S_n are empty. Therefore any finitely additive measure on Σ^+ is c.a. by default. Therefore $\bar{\lambda}, \bar{\mu}$ are c.a., $\bar{\lambda}$ vanishes on $\bar{\mu}$ null sets, but $\bar{\lambda} \not\ll \bar{\mu}$, since

$$\bar{\lambda} \ll \bar{\mu} \iff \lambda \ll \mu$$

(If $\lambda \ll \mu$, then $\lambda \in L_1(\mu)$ by Radon-Nikodym theorem)

(Bartle-Dunford-Schwartz)

THEOREM: A uniformly bounded family $\{F_\tau : \tau \in T\}$ of c.a. measures is unif. s.a. iff unif. c.a. iff $\exists$ c.a. non-negative (finite) measure μ on Σ s.t.

$$\lim_{\mu(E) \to 0} F_\tau(E) = 0 \quad \text{unif. in } \tau \in T$$

Proof. unif. c.a. $\iff$ unif. s.a.

Suppose $\exists$ such a c.a. μ on Σ. Let (E_n) be a disjoint sequence. Then

$$\mu\left(\bigcup_{n=j}^{\infty} E_n\right) \longrightarrow 0$$

as $j \to \infty$. Then

$$0 = \lim_j \sup_\tau \| F_\tau(\bigcup_{n=j}^\infty E_n) \| \overset{c.a.}{=} \lim_j \sup_\tau \| \sum_{n=j}^\infty F_\tau(E_n) \|$$

This proves unif. s.a.

Now suppose unif. s.a. Then $\{x^* F_\tau : \tau \in T, \|x^*\| \le 1\}$ is unif. s.a. So it suffices to prove result for family $\{\mu_\tau : \tau \in T\}$ of scalar (signed) measures.

Claim: For each $\varepsilon > 0$ $\exists\, \tau_1, \dots, \tau_n$ s.t.

$$\sup_{1 \le i \le n} |\mu_{\tau_i}|(E) = 0 \implies \sup_{\tau \in T} |\mu_\tau(E)| < \varepsilon$$

Suppose not. Pick $\tau_1 \in T$ arbitrarily. Then $\exists\, E_1 \in \Sigma$ and $\tau_2 \in T$ s.t.

$$|\mu_{\tau_1}|(E_1) = 0 \text{ and } |\mu_{\tau_2}(E_1)| \ge \varepsilon$$

$\exists\, E_2 \in \Sigma$, $\exists\, \tau_3 \in T$ s.t.

$$|\mu_{\tau_1}|(E_2) = |\mu_{\tau_2}|(E_2) = 0 \text{ and } |\mu_{\tau_3}(E_2)| \ge \varepsilon$$

Iterate to produce (E_n), τ_n with

$$\sup_{1 \le i \le n} |\mu_i|(E_n) = 0, \quad |\mu_{\tau_{n+1}}(E_n)| \ge \varepsilon \qquad (*)$$

Set $H_n = \bigcup_{j=n}^\infty E_j$. Observe $H_n \downarrow$. By uniform countable additivity

$$\lim_n |\mu_\tau(H_n)|$$

exists uniformly for $\tau \in T$. Hence

$$\lim_n |\mu_{\tau_k}(H_n)|$$

exists uniformly for k. But for each k, by (*)

$$\lim_n \mu_{\tau_k}(H_n) = 0$$

But

$$\mu_{\tau_{n+1}}(H_n) = \mu_{\tau_{n+1}}(E_n) + \underbrace{0}_{(*)}$$

$$\Rightarrow |\mu_{\tau_{n+1}}(H_n)| = |\mu_{\tau_{n+1}}(E_n)| \geq \varepsilon$$

4/3 VECTOR MEASURES

We have shown that $\forall\, m \in \mathbb{N}$ $\exists\, \tau_1^m, \ldots, \tau_{n_m}^m$ and $\mu_{\tau_1^m}, \ldots, \mu_{\tau_{n_m}^m}$ in the family s.t.

$$\sup_{1 \le j \le n_m} |\mu_{\tau_j^m}|(E) = 0 \Rightarrow \sup_{\tau \in T} |\mu_\tau(E)| \le \frac{1}{m}$$

Set

$$\lambda_m := \frac{1}{n_m} \sum_{j=1}^{n_m} |\mu_{\tau_j^m}|$$

Observe

$$0 \le \lambda_m(E) \le \sup_{\tau \in T} |\mu_\tau|(E)$$

$$\lambda_m(E) = 0 \Rightarrow \sup_{\tau \in T} |\mu_\tau(E)| \le \frac{1}{m}$$

Set

$$\mu := \sum_{m=1}^{\infty} \frac{1}{2^m} \lambda_m$$

Then

$$\mu(E) = 0 \Rightarrow \sup_{\tau \in T} |\mu_\tau(E)| = 0$$

Notice that

$$(*) \qquad 0 \le \mu(E) \le \sup_{\tau} |\mu_\tau|(E) \qquad \forall E$$

Define $F : \Sigma \longrightarrow \ell_\infty(T)$ by

$$F(E)(\tau) = \mu_\tau(E)$$

(unif. boundedness important here to make sure we map into $\ell_\infty(T)$). Notice that

$$\mu(E) = 0 \Rightarrow F(E) = 0$$

Uniform c.a. for (μ_τ) implies that F is countably additive. By Pettis's theorem, $F << \mu$, i.e.

$$\lim_{\mu(E) \to 0} \|F(E)\| = 0$$

$$\Longrightarrow \lim_{\mu(E) \to 0} \sup_{\tau} |\mu_\tau(E)| = 0$$

$\blacksquare$

__Corollary__: Let $F : \Sigma \to X$ be a c.a. vector measure. Then there exist a c.a. finite non-negative measure μ on Σ s.t. $F << \mu$. In fact μ may be selected s.t.

$$\forall E \qquad 0 \le \mu(E) \le \|F\|(E)$$

Proof. c.a. on σ-field $\Rightarrow$ s.a. $\Rightarrow$ F has bounded range. Consider $\{x^*F : \|x^*\| \le 1\}$. c.a. of F implies that this family is unif. c.a. Apply last theorem to get μ s.t. $x^*F \ll \mu$ unif. in $\|x^*\| \le 1$. Hence

$$\lim_{\mu(E)\to 0} \|F(E)\| = \lim_{\mu(E)\to 0} \sup_{\|x^*\|\le 1} |x^*F(E)| = 0$$

and (*) of last theorem gives

$$0 \le \mu(E) \le \|F\|(E).$$

∎

(p267)

Rybakov's theorem (chapter 9) gives a μ of the form $\mu = |x^*F|$ for some $\|x^*\| \le 1$.

➡ <u>COROLLARY</u>: Let $F: \Sigma \to X$ be c.a. Then $F(E)$ is relatively weakly compact

Proof. Choose the μ guaranteed above for F. Define $T: L_\infty(\mu) \to X$ by

$$T(f) := \int_\Omega f \, dF$$

(Need here that F vanishes on μ-null sets to make well defined). Then T

is a bounded linear operator. If we can show that T is weakly compact then we'll be done since

$$F(\Sigma) \subset T(\text{unit ball of } L_\infty(\mu))$$

To show T is weakly compact, we'll show T is w^*-w continuous. Let $(f_\beta) \subset L_\infty(\mu)$ s.t. $f_\beta \to f$ weak*, $f \in L_\infty(\mu)$. Let $x^* \in X^*$ and set

$$g_{x^*} = \frac{d\, x^* F}{d\mu} \in L_1(\mu)$$

Then

$$\begin{aligned} x^* T(f_\beta) &= x^* \int_\Omega f_\beta \, dF = \int_\Omega f_\beta \, d(x^* F) \\ &= \int_\Omega f_\beta \, g_{x^*} \, d\mu \longrightarrow \int_\Omega f g_{x^*} \, d\mu \\ &= \int_\Omega f \, d(x^* F) \\ &= x^* \int_\Omega f \, dF = x^* T(f) \end{aligned}$$

and so T is weak*-to-weak continuous.

$\boxtimes$

HW/ 1) Prove that w^*-w continuous operators from $L_\infty(\mu)$ to X correspond precisely to the c.a. μ-cont $F: \Sigma \to X$ (Orlicz-Pettis th^m)

[**DEFINITION:** a subset of a Banach space has the Banach-Saks property if every sequence in A has a subsequence whose arithmetic averages converge in norm.]

2) The range of a c.a. vector measure has the B-S property (Diestel-Seifert)

Outline of proof – Let $T(f) = \int f\, dF$. Set $T: L_\infty(\mu) \to X$ good measure assoc. with F

$$P = \{ f \in L_\infty(\mu) : 0 \leq f \leq 1 \quad a.e. \}$$

① Prove if $(f_n) \subset P$ s.t. $f_n \to f \in P$ in L_2 norm, then $T(f_n) \to T(f)$ (Egoroff)

② Prove $L_2(\mu)$ has B-S prop (Riesz-Nagy p.80)

③ Prove $T(P)$ has B-S prop

④ Deduce $F(\Sigma)$ has B-S prop.

Nikodym Boundedness Theorem

Let $\{F_\tau\}$ be a family of bounded finitely additive vector measure on a σ-field Σ s.t.

$$\sup_{\tau\in T} \|F_\tau(E)\| < \infty$$

for every $E \in \Sigma$. Then

$$\sup_{E\in\Sigma}\ \sup_{\tau\in T} \|F_\tau(E)\| < \infty$$

(i.e. the ranges of the F_τ's are uniformly bounded)

Corollary: Let (T_α) be a family of bounded linear operators from $L_\infty(\mu)$ to X s.t. $\sup_\alpha \|T_\alpha(\chi_E)\| < \infty$ $\forall E \in \Sigma$. Then $\sup_\alpha \|T_\alpha\| < \infty$.

Proof of corollary. Let $F_\alpha(E) := T_\alpha(\chi_E)$. Nikodym Boundedness implies F_α's have unif. bdd range, and so unif. bdd. semivariation. Thus the operators have unif. bdd. norm since $\|T_\alpha\| = \|F_\alpha\|$.

Proof of theorem. Enough to show $\{x^* F_\tau : \|x^*\| \le 1, \tau\in T\}$ has unif. bdd, so it suffices to prove it for a scaler (signed) f.a. measures. Enough to prove it for a sequence (μ_n) of bounded finitely additive measures on Σ

(Compare with proof for c.a. case in Dunford-Schwartz)

Let (μ_n) be such a sequence. Assume $\sup_n |\mu_n(E)| < \infty \ \forall E$ but

$$\sup_n \sup_{E \in \Sigma} |\mu_n(E)| = +\infty$$

General principle: Let $\rho > 0$. We can find $n, E \in \Sigma$ such that

$$|\mu_n(E)| \geq \sup_k |\mu_k(\Omega)| + \rho$$

Observe that

$$\begin{aligned} |\mu_n(\Omega \setminus E)| &\geq |\mu_n(E)| - |\mu_n(\Omega)| \\ &\geq |\mu_n(E)| - \sup_k |\mu_k(\Omega)| \\ &\geq \rho \end{aligned}$$

i.e. $|\mu_n(E)| \geq \rho$ and $|\mu_n(\Omega \setminus E)| \geq \rho$.

Let n_1 be the smallest positive integer s.t. $\exists\, E_1 \in \Sigma$ with

$$|\mu_{n_1}(E_1)| \geq 2 \ , \quad |\mu_{n_1}(F_1)| \geq 2$$

(where $F_1 = \Omega \setminus E_1$). At least one of the following is true:

$$\sup_n \sup_E |\mu_n(E \cap E_1)| = +\infty \quad \text{or} \quad \sup_n \sup_E |\mu_n(E \cap F_1)| = +\infty$$

In the first case set $S_1 = E_1$ and $T_1 = F_1$. Otherwise set $S_1 = F_1, T_1 = E_1$. Either way

$$|\mu_{n_1}(S_1)| \geq 2 \ , \ |\mu_{n_1}(T_1)| \geq 2$$

There exist smallest $n_2 > n_1$ s.t. $\exists E_2^{\subset S_1}, F_2$ with $F_2 = S_1 \setminus E_2$ satisfying

$$|\mu_{n_2}(E_2)| \text{ and } |\mu_{n_2}(F_2)| > 3 + |\mu_{n_2}(T_1)|$$

At least one of

$$\sup_n \sup_E |\mu_n(E \cap E_2)| = +\infty \quad \text{or} \quad \sup_n \sup_E |\mu_n(E \cap F_2)| = +\infty$$

$$S_2 = E_2 \quad T_2 = F_2 \qquad\qquad \text{otherwise} \qquad S_2 = F_2 \quad T_2 = E_2$$

Continue this process to obtain a disjoint sequence (T_n) and $n_1 < n_2 < n_3 \cdots$ s.t.

$$|\mu_{n_k}(T_k)| > \sum_{j=1}^{k-1} |\mu_{n_k}(T_j)| + k + 1 \qquad \text{for } k \geq 2$$

Relabel $n_k \to k$, so

$$(*) \qquad |\mu_k(T_k)| > \sum_{j=1}^{k-1} |\mu_k(T_j)| + k + 1$$

Partition $\mathbb{N}$ into infinitely many disjoint infinite subsets $N_1, N_2, \ldots$ By finite additivity we have

#1a

Choose smallest $n_2 > n_1$ s.t. $\exists E$ with

$$|\mu_{n_2}(E \cap S_1)| > \underbrace{3 + \sup_k |\mu_k(T_1)|}_{\rho} + \sup_k |\mu_k(S_1)|$$

Let $E_2 = E \cap S_1$. Then

$$|\mu_{n_2}(E_2)| > \rho > 3 + |\mu_{n_2}(T_1)|$$
$$|\mu_{n_2}(S_1 \setminus E_2)| > \rho > 3 + |\mu_{n_2}(T_1)|$$

$$\sum_{n=1}^{\infty} |\mu_1|\Big(\bigcup_{j\in N_n} T_j\Big) \le |\mu_1|\Big(\bigcup_j T_j\Big) \le |\mu_1|(\Omega)$$

Hence $\exists\, n$ s.t. $1 \notin N_n$ and

$$|\mu_1|\Big(\bigcup_{j\in N_n} T_j\Big) < 1$$

i.e. there is a subsequence (T_{k_i}) of $(T_i)_{i\ge 2}$ s.t.

$$|\mu_1|\Big(\bigcup_{i=1}^{\infty} T_{k_i}\Big) < 1$$

Repeat this argument with $(T_i)_{i\ge 2}$ replaced by $(T_{k_i})_{i\ge 2}$ and $|\mu_1|$ replaced by $|\mu_{k_1}|$

	T_1	T_2	T_3	- - -	- - -	- - -
$\lvert\mu_1\rvert<1$ on		T_{k_1}		T_{k_2}	- - -	- - -
$\lvert\mu_{k_1}\rvert<1$ on				$T_{k_{i_1}}$	- - -	- - -
$\lvert\mu_{k_{i_1}}\rvert<1$ on					$T_{k_{i_{j_1}}}$	- - -

Relabel again

$$T_1 \quad T_2 \quad T_3$$

$|\mu_1| < 1$ $\qquad\qquad T_{m_1}$

$|\mu_{m_1}| < 1$ $\qquad\qquad\qquad T_{m_2}$

$|\mu_{m_2}| < 1$ $\qquad\qquad\qquad\qquad$ first terms

$$1 < m_1 < m_2 < m_3$$

By monotonicity

$$(**) \qquad |\mu_{m_j}|\left(\bigcup_{i=j+1}^{\infty} T_{m_i}\right) < 1$$

Let

$$D = \bigcup_{i=1}^{\infty} T_{m_i}$$

Claim: $\sup_k |\mu_k(D)| = +\infty$ ⚡!

To this end, finite additivity gives

$$|\mu_{m_j}(D)| = \left|\mu_{m_j}\left(\bigcup_{i<j} T_{m_i}\right) + \mu_{m_j}(T_{m_j}) + \mu_{m_j}\left(\bigcup_{i>j} T_{m_i}\right)\right|$$

$$\geq |\mu_{m_j}(T_{m_j})| - \sum_{i<j} |\mu_{m_j}(T_{m_i})| - |\mu_{m_j}|\left(\bigcup_{i>j} T_{m_i}\right)$$

(*)

(**)

$$\geq \sum_{p=1}^{m_j - 1} |\mu_{m_j}(T_p)| + m_j + 1 - \sum_{i<j} |\mu_{m_j}(T_{m_i})| - 1$$

original T_p's

$$\geq m_j \longrightarrow \infty \text{ as } j \to \infty$$

$$\left(\text{since } \sum_{i<j} |\mu_{m_j}(T_{m_i})| \leq \sum_{p=1}^{m_j - 1} |\mu_{m_j}(T_p)| \right)$$

$\blacksquare$

4/5 VECTOR MEASURES

<u>COROLLARY</u> (Dieudonné - Grothendieck) Let $F : \Sigma \to X$ be a finitely additive vector measure s.t. there exists a total subset $\Gamma \subset X^*$ with the property that the measure x^*F is bounded $\forall x^* \in \Gamma$. Then F is bounded.

Proof. Since $\|F(E)\| = \sup_{\|x^*\| \le 1} |x^*F(E)|$ is finite, it is enough to prove x^*F is bounded for each $x^* \in X^*$. For then the Nikodym boundedness theorem will apply to the family $\{x^*F : \|x^*\| \le 1\}$ to give

$$\sup_E \sup_{\|x^*\| \le 1} |x^*F(E)| < \infty$$

$$\Rightarrow \sup_E \|F(E)\| < \infty$$

i.e. F has bounded range.

Let $\mathcal{M} = \{x^* \in X^* : x^*F \text{ is bounded measure}\}$. $\mathcal{M}$ is a weak* dense subspace of X^* since $\Gamma \subset \mathcal{M}$. Let

$$\mathcal{M}_1 = \{x^* \in \mathcal{M} : \|x^*\| \le 1\}$$

If we can show $\mathcal{M}_1$ is w* closed, then by Krein - Smulian we would have $\mathcal{M}$ being weak* closed. To show $\mathcal{M}_1$ is weak*-closed, first note that

$$\sup_{x^* \in \mathfrak{M}_1} |x^*F(E)| \leq \|F(E)\| < \infty$$

for every $E \in \Sigma$. By Nikodym boundedness,

$$\sup_{E \in \Sigma} \sup_{x^* \in \mathfrak{M}_1} |x^*F(E)| \leq K < \infty$$

Now let $x^* \in$ weak* closure of $\mathfrak{M}_1$. Choose a net $(x^*_\alpha) \subset \mathfrak{M}_1$ such that

$$x^*_\alpha \longrightarrow x^* \quad \text{weak}^*$$

Then for $E \in \Sigma$ we have

$$|x^*F(E)| = \lim_\alpha |x^*_\alpha F(E)| \leq K$$

Hence x^*F is a bounded measure and so $x^* \in \mathfrak{M}_1$. Therefore $\mathfrak{M}_1$ is weak*-closed.

$\boxtimes$

<u>COROLLARY</u>: (Seever) Let $T: X \longrightarrow B(\Sigma)$ be a bounded linear operator s.t.

$$T(X) \supset \{\chi_E : E \in \Sigma\}$$

Then T is onto $B(\Sigma)$.

Proof. Obviously $T(X)$ is dense in $B(\Sigma)$. Therefore it suffices to show T has a closed range, so it is enough to show T^* has a closed range.

Consider $T^*: B(\Sigma)^* \longrightarrow X^*$. $B(\Sigma)^*$ is the space of all bounded finitely additive scaler measures on Σ with bounded variation. Suppose $x^* \in X^*$ is not in the range of T^*, but is a cluster point of the range of T^*. Find a sequence (μ_n) in $B(\Sigma)^*$ s.t.

$$T^*(\mu_n) \longrightarrow x^*$$

and observe that

$$\overline{\lim} \, |\mu_n| = +\infty$$

For, if not, then $\sup |\mu_n| < \infty$, so (μ_n) has a weak* convergent subnet $(\mu_{n_\alpha}) \longrightarrow \mu \in B(\Sigma)^*$. But since T^* is an adjoint operator, it is weak*-weak* continuous, so

$$x^* = \lim_\alpha T^*(\mu_{n_\alpha}) \overset{w^*}{=} T^*(\mu) \quad \lightning$$

WLOG $T^*(\mu_n) \longrightarrow x^*$ but $|\mu_n| \longrightarrow \infty$ (pass to subsequence if necessary)

Fix $E \in \Sigma$. By hypothesis $\exists x \in X$ such that $Tx_E = \chi_E$. Consider

$$\underset{\text{measure}}{\underbrace{|\mu_n(E)|}} = \underset{\text{linear functional}}{\underbrace{|\mu_n T x_E|}} = |T^*(\mu_n)(x_E)| \longrightarrow |x^*(x_E)|$$

Therefore $\sup_n |\mu_n(E)| < K_E < \infty$. Hence

$$\sup_E \sup_n |\mu_n(E)| < \infty$$

which contradicts that $|\mu_n| \longrightarrow \infty$.
Hence the range is norm closed.

$\blacksquare$

<u>ROSENTHAL'S LEMMA</u>: Let (μ_n) be a uniformly bounded sequence of finitely additive scalar measures on a field $\mathcal{F}$. Let (E_n) be a disjoint sequence in $\mathcal{F}$. Let $\varepsilon > 0$. Then there exists a subsequence $n_1 < n_2 < n_3 < \dots$ s.t.

$$|\mu_{n_j}|\Big(\bigcup_{\substack{i \neq j \\ i \in \Delta}} E_{n_i}\Big) < \varepsilon$$

for all finite subsets Δ of $\mathbb{N}$. If $\mathcal{F}$ is a σ-field, then $n_1 < n_2 < \dots$ may be chosen s.t.

$$|\mu_{n_j}|\Big(\bigcup_{i \neq j} E_{n_i}\Big) < \varepsilon.$$

Proof: (σ-field case) WLOG $\sup_n |\mu_n|(\Omega) \leq 1$. Partition $\mathbb{N}$ into infinitely many disjoint infinite subsets M_j. If there exists p such that there exists no $k \in M_p$ with

<u>SUCCESS</u>

$$|\mu_k|\Big(\bigcup_{\substack{j\neq k \\ j\in M_p}} E_j\Big) \geq \varepsilon$$

then we're done. Just list $M_p = \{n_1 < n_2 < n_3 < \dots\}$. So assume no such p exists. Then $\forall p\ \exists\, k_p \in M_p$ with

$$|\mu_{k_p}|\Big(\bigcup_{\substack{j\neq k_p \\ j\in M_p}} E_j\Big) \geq \varepsilon$$

Notice

$$|\mu_{k_p}|\Big(\bigcup_{q=1}^{\infty} E_{k_q}\Big) + |\mu_{k_p}|\Big(\bigcup_{n=1}^{\infty} E_n \setminus \bigcup_{n=1}^{\infty} E_{k_n}\Big) \leq 1$$

for all p. But

$$\bigcup_{n=1}^{\infty} E_n \setminus \bigcup_{n=1}^{\infty} E_{k_n} \supseteq \bigcup_{\substack{j\in M_p \\ j\neq k_p}} E_j$$

Therefore

$$|\mu_{k_p}|\Big(\bigcup_{q=1}^{\infty} E_{k_q}\Big) + \varepsilon \leq 1$$

Replace Ω by $\bigcup_{q=1}^{\infty} E_{k_q}$ and replace (E_n) by (E_{k_n}). Do the same thing again. If successful, stop. If not successful, proceed to get a further subsequence s.t.

$$|\mu_{k_{a_p}}|\left(\bigcup_{p=1}^{\infty} E_{k_{a_p}}\right) \leq 1 - 2\varepsilon$$

Do this again and again. Evidently we will be successful before n steps provided $1 - n\varepsilon < 0$.

(Field case) Let $\Sigma = \sigma(\mathcal{F})$. Then $B(\mathcal{F})$ is a closed subspace of $B(\Sigma)$. Given (μ_n) and (E_n), view μ_n as a sequence in $B(\mathcal{F})^*$. For each n let $\bar{\mu}_n$ be a Hahn-Banach norm preserving extension of μ_n to a member of $B(\Sigma)^*$. By the σ-field case there exist $n_1 < n_2 < \cdots$ s.t.

$$|\bar{\mu}_{n_j}|\left(\bigcup_{i \neq j} E_{n_i}\right) < \varepsilon$$

Hence, if Δ is a finite subset of $\mathbb{N}$, then

$$|\mu_{n_j}|\left(\bigcup_{\substack{i \neq j \\ i \in \Delta}} E_{n_i}\right) \leq |\bar{\mu}_{n_j}|\left(\bigcup_{\substack{i \neq j \\ i \in \Delta}} E_{n_i}\right)$$

$\mu_n, \bar{\mu}_n$ agree on $\mathcal{F}$

$$\leq |\bar{\mu}_{n_j}|\left(\bigcup_{i \neq j} E_{n_i}\right) < \varepsilon$$

$\blacksquare$

THEOREM: (Diestel-Faires). Suppose $G : \mathcal{F} \to X$ is a bounded vector measure but is not strongly additive. Then there exist a disjoint sequence (E_n) and an isomorphism $T : c_0 \to X$ s.t.

$$T(e_n) = G(E_n)$$

If $\mathcal{F}$ is a σ-field, then we can find disjoint sequence (E_n) and an isomorphism $T : \ell_\infty \to X$ s.t.

$$T(\chi_S) = G\Big(\bigcup_{n \in S} E_n \Big)$$

for all $S \subset \mathbb{N}$.

Proof. (Field case). Suppose $G : \mathcal{F} \to X$ is not s.a. Then $\exists$ disjoint sequence (E_n) s.t.

$$\lim_n \|G(E_n)\| \neq 0$$

Pass to a subsequence (if necessary) to arrange

$$\|G(E_n)\| > \varepsilon > 0 \quad \forall n$$

for some $\varepsilon > 0$. Choose for each n $x_n^* \in X^*$ s.t. $\|x_n^*\| = 1$ and

$$x_n^* G(E_n) > \varepsilon \quad \forall n$$

Since G is bounded and $\|x_n^*\| \leq 1$, the seq $(x_n^* G)$ is uniformly bounded in variation. Apply Rosenthal's lemma to get $n_1 < n_2 < \cdots$

$$|x^*_{n_j} G| \Big(\bigcup_{\substack{i \neq j \\ i \in \Delta}} E_{n_i} \Big) < \varepsilon/2$$

for all finite $\Delta \subset \mathbb{N}$. (Remember $x^*_{n_j} G(E_{n_j}) > \varepsilon$). Define $T : c_0 \to X$ by

$$T((\alpha_n)) := \sum_{j=1}^{\infty} \alpha_j G(E_{n_j})$$

finitely non-zero

Notice for $\|x^*\| \leq 1$

$$|x^* T((\alpha_n))| \leq \sum_{j=1}^{\infty} |\alpha_j x^* G(E_{n_j})|$$

$$\leq \|(\alpha_n)\|_{c_0} \sum_{j=1}^{\infty} |x^* G(E_{n_j})|$$

$$\leq \|(\alpha_n)\|_{c_0} |x^* G|(\Omega)$$

$$\leq \|(\alpha_n)\|_{c_0} \|G\|(\Omega) < \infty$$

T is therefore continuous, linear, densely defined, and hence has a continuous extension to all of c_0. By necessity

$$T(\alpha_n) = \sum_{j=1}^{\infty} \alpha_j G(E_{n_j})$$

$$\left[\sum |x^*(x_n)| < \infty \ \forall x^* \forall n \implies \sum_{n=1}^{\infty} \alpha_n x_n \text{ converges } \forall (\alpha_n) \in c_0 \right]$$

4/10 VECTOR MEASURES

Proof (cont.)

To see that T^{-1} exist, let $(\alpha_j) \in c_0$ and consider $x^*_{n_j} T(\alpha_n)$
We get

$$|x^*_{n_j} T(\alpha_i)| = \left| \sum_{i=1}^{\infty} x^*_{n_j} \alpha_i G(E_{n_i}) \right|$$

$$\geq |x^*_{n_j} \alpha_j G(E_{n_j})| - \lim_m \sum_{\substack{i=1 \\ i\neq j}}^{m} |x^*_{n_j} \alpha_i G(E_{n_i})|$$

$$> |\alpha_j| \varepsilon - \|(\alpha_i)\|_{c_0} \lim_m \sum_{\substack{i=1 \\ i\neq j}}^{m} |x^*_{n_j} G(E_{n_i})|$$

$$\geq |\alpha_j| \delta - \|(\alpha_i)\|_{c_0} \lim_m |x^*_{n_j} G| \left(\bigcup_{\substack{i=1 \\ i\neq j}}^{m} E_{n_i} \right)$$

$$\geq |\alpha_j| \delta - \|(\alpha_i)\|_{c_0} \varepsilon/2$$

Therefore

$$\|T(\alpha_i)\| \geq \sup_j |x^*_{n_j} T(\alpha_i)|$$

$$\geq \varepsilon \|(\alpha_i)\|_{c_0} - \frac{\varepsilon}{2} \|(\alpha_i)\|_{c_0} = \frac{\varepsilon}{2} \|(\alpha_i)\|_{c_0}$$

Thus continuous inverse. $\blacksquare$

This proves that a bounded $G : \mathcal{F} \longrightarrow X$ which is not strongly additive implies $G(\mathcal{F})$ contains an isomorphic image of the c_0 unit vector basis.

Consider the σ-field case:

$$G : \Sigma \longrightarrow X$$

G bounded but not strongly additive. We use Rosenthal's lemma to get

$$|x^*_{n_j} G(E_{n_j})| > \delta \qquad |x^*_{n_j} G| \left(\bigcup_{i \neq j} E_{n_i} \right) < \delta/2$$

with $x^*_{n_j}$ as before. To define $T : l_\infty \longrightarrow X$, take a finitely valued sequence (= simple function) in l_∞ which we write as

$$\sum_{i=1}^{n} \beta_i \chi_{S_i}$$

$\cup S_i = \mathbb{N}$

$S_i \subseteq \mathbb{N}$ are disjoint. Define

need σ-field here

$$T\left(\sum_{i=1}^{n} \beta_i \chi_{S_i} \right) = \sum_{i=1}^{n} \beta_i \, G\left(\bigcup_{j \in S_i} E_{n_j} \right)$$

Easy to see $\|T\| \leq \|G\|$. Thus has continuous extension to l_∞.

Now look at

$$(*) \quad \left| x^*_{n_j} T\left(\sum_{i=1}^{n} \beta_i \chi_{S_i} \right) \right| = \left| \sum_{i=1}^{n} \beta_i x^*_{n_j} G\left(\bigcup_{k \in S_i} E_{n_k} \right) \right|$$

Let i_p be the i s.t. $n_j \in S_{i_p}$. Then we have

$$(*) \geq |\beta_{i_p} x^*_{n_j} G(E_{n_j})| - \sum_{i \neq i_p} |\beta_i x^*_{n_j} G(\bigcup_{k \in S_i} E_{n_k})|$$

$$- |\beta_{i_p} x^*_{n_j} G(\bigcup_{\substack{k \in S_{i_p} \\ n_k \neq n_j}} E_{n_k})$$

$$\geq |\beta_{i_p}| \delta - \| \sum_{i=1}^{n} \beta_i \chi_{S_i} \|_\infty \, |x^*_{n_j} G| (\bigcup_{i \neq j} E_{n_i})$$

$$\geq |\beta_{i_p}| \delta - \max |\beta_i| \, \delta/2$$

Therefore

$$\| T(\sum \beta_i \chi_{S_i}) \| \geq \sup_j |x^*_{n_j} T(\sum \beta_i \chi_{S_i})|$$

$$\geq \sup_p |\beta_{i_p}| \delta - \| \sum \beta_i \chi_{S_i} \|_\infty \, \delta/2$$

$$= \delta/2 \, \| \sum \beta_i \chi_{S_i} \|_\infty$$

Therefore $\|T(x)\| \geq \delta/2 \, \|x\|_\infty \ \forall x \in l_\infty$, and so T^{-1} is continuous.
Since

$$\chi_S \xrightarrow{T} G(\bigcup_{i \in S} E_{n_i})$$

$G(\Sigma)$ contains the isomorphic image of all 0-1 valued l_∞ members. ∎

<u>COROLLARY</u>: $l_\infty \not\hookrightarrow X$, $G : \Sigma \longrightarrow X$ bounded, then G is strongly additive.

(What kind of field can you get by with to insure s.a. in above?)

<u>COROLLARY</u>: Let $G : \mathcal{F} \longrightarrow X$ be a bounded vector measure. Then G is strongly additive iff $\lim G(E_n)$ exists weakly for all monotone increasing sequences (E_n) in Σ.

Proof. Suppose the weak limit exists and G is not s.a. Let (B_n) be a disjoint sequence in $\mathcal{F}$ s.t. $\exists$ isomorphism $T : c_0 \longrightarrow X$ s.t.

$$T(e_n) = G(B_n)$$

Now

$$T\left(\sum_{n=1}^{m} e_n\right) = \underbrace{G\left(\bigcup_{n=1}^{m} B_n\right)}$$

Isomorphism

this limit does not exist weakly in c_0

limit exists weakly by hypothesis

∎

Corollary: (Orlicz-Pettis) Let $\sum x_n$ be a formal series in X s.t. every subsequence converges weakly. Then $\sum x_n$ is convergent.

Proof. For $A \subset \mathcal{P}(\mathbb{N})$ set

$$G(A) := \text{weak} \sum_{n \in A} x_n$$

First notice that $G(\mathcal{P}(\mathbb{N}))$ is separable. Also notice

$$\sum |x^*(x_n)| < \infty$$

for every $x^* \in X^*$. Define $T : X^* \longrightarrow \ell_1$ by

$$T(x^*) = (x^*(x_n))$$

By closed graph theorem T is continuous. Therefore

$$\|G\|(\mathbb{N}) = \sup_{\|x^*\| \leq 1} \sum |x^*(x_n)| < \infty$$

Hence G is bounded. Since $G(\mathcal{P}(\mathbb{N}))$ is separable, G is s.a. (can't contain all 0-1 valued sequences). Therefore

$$\sum_{n=1}^{\infty} G(\{n\}) = \sum x_n$$

is convergent.

$\boxtimes$

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$\forall A \qquad G(A) \in$ weak closure of $sp\{x_n : n \in \mathbb{N}\} = \underbrace{\overline{sp}\{x_n : n \in \mathbb{N}\}}_{\text{separable}}$

$$\sum |x^*(x_n)| = \sum_A x^*(x_n) - \sum_B x^*(x_n)$$

$$A = \{n : x^*(x_n) \geq 0\}$$
$$B = \{n : x^*(x_n) < 0\}$$

Suppose $x_m^* \to x^*$ and $Tx_m^* \to \beta$. Then

$$\beta_n = \lim_{m\to\infty} (Tx_m^*)_n = \lim_{m\to\infty} x_m^*(x_n) = x^*(x_n) = T(x^*)_n$$

and so $\beta = T(x^*)$.

$$\|G\|(\mathbb{N}) = \sup_{\|x^*\|\leq 1} |x^*G|(\mathbb{N}) = \sup_{\|x^*\|\leq 1} \sup_{\pi} \sum_{A\in\pi} |x^*G(A)|$$
$$= \sup_{\|x^*\|\leq 1} \sup_{\pi} \sum_{A\in\pi} \Big| \sum_{n\in A} x^* x_n \Big|$$
$$\leq \sup_{\|x^*\|\leq 1} \sup_{\pi} \sum_{n=1}^{\infty} |x^* x_n| = \sup_{\|x^*\|\leq 1} \sum_{n=1}^{\infty} |x^* x_n| = \|T\| < \infty$$

<u>COROLLARY</u>: (Bessaga - Pelczynski) $c_0 \not\hookrightarrow X$ iff $\sum x_n$ converges in norm whenever $\sum |x^*(x_n)| < \infty \ \forall x^* \in X^*$.

$$\left[\ c_0 \not\hookrightarrow X \iff \text{all } \underbrace{\text{WUC's}}_{\text{weakly unconditionally convergent}} \text{ are UC's}\ \right]$$

Proof. Let $c_0 \xhookrightarrow{T} X$. $\sum e_n$ is a WUC but not a UC. Therefore $\sum T(e_n)$ is a WUC but not a UC.

Assume $c_0 \not\hookrightarrow X$. Let $\mathcal{F}$ = field consisting of the finite subsets of $\mathbb{N}$ and their complements. Let $\sum x_n$ be a WUC

$$G(E) := \begin{cases} \sum\limits_{n\in E} x_n & E \text{ finite} \\ -\sum\limits_{n\in \mathbb{N}\setminus E} x_n & \mathbb{N}\setminus E \text{ finite} \end{cases}$$

Then G is finitely additive, and since $\sum x_n$ is a WUC, G is bounded. Since $c_0 \not\hookrightarrow X$, G is s.a., and therefore

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} G(\{n\})$$

is convergent.

$\boxtimes$

<u>COROLLARY</u>: (Bessaga - Pelczynski) $c_0 \hookrightarrow X^* \Rightarrow l_\infty \hookrightarrow X^*$

Proof. Let $\sum x_n^*$ be a WUC in X^* that does not converge.

Put for $A \subset \mathbb{N}$

$$G(A) = \text{weak}^* \sum_{n \in A} x_n^*$$

Then

$$\sup_{\|x\| \leq 1} |xG|(\mathbb{N}) < \infty$$

and $G : \mathcal{P}(\mathbb{N}) \longrightarrow X^*$ is not s.a. since $\sum_{n=1}^{\infty} G\{n\} = \sum x_n^*$ is not convergent. G bounded, so $l_\infty \hookrightarrow X^*$.

$\blacksquare$

(1971 Israeli J.)

<u>COROLLARY</u>: (Kalton) Let $l_\infty \not\hookrightarrow X$ and let $\sum x_n$ be a (formal) series in X s.t. there exists a total subset Γ of X^* with the property that every subseries of $\sum x_n$ is $\sigma(X, \Gamma)$ convergent, i.e. $\forall A \subset \mathbb{N}$ $\exists x_A \in X$ s.t.

$$\sum_{n \in A} x^*(x_n) = x^*(x_A)$$

for all $x^* \in \Gamma$. Then $\sum x_n$ is norm convergent.

Proof. For $A \in \mathcal{P}(\mathbb{N})$ set $G(A) = x_A$. Then x^*G is a bounded scalar measure for all $x^* \in \Gamma$. By Dieudonne-Grothendieck, G is a bounded vector measure. $l_\infty \not\hookrightarrow X \Rightarrow G$ is s.a. $\Rightarrow$

$$\sum x_n = \sum G(\{n\}) \text{ is convergent}$$

$\blacksquare$

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X^* is weak* sequentially complete and it is easy to see that the partial sums are weak*-Cauchy

Define $T : X \to \ell_1$ by $Tx = (x_n^* x)$. Then T is bounded (closed graph theorem). For each $E \subset \mathbb{N}$

$$\|G(E)\| = \sup_{\|x\| \leq 1} |G(E)x| = \sup_{\|x\| \leq 1} \left| \sum_{n \in E} x_n^* x \right|$$

$$\leq \sup_{\|x\| \leq 1} \sum_{n=1}^{\infty} |x_n^* x| = \|T\|$$

Therefore $\|G\|(\mathbb{N}) \leq 4 \sup_{E \subset \mathbb{N}} \|G(E)\| \leq 4\|T\| < \infty$

Let $x^* \in \Gamma$. Then

$$|x^* G|(\mathbb{N}) = \sup_{\pi} \sum_{A \in \pi} |x^* G(A)| = \sup_{\pi} \sum_{A \in \pi} \left| \sum_{n \in A} x^*(x_n) \right|$$

$$\leq \sup_{\pi} \sum_{A \in \pi} \sum_{n \in A} |x^*(x_n)| = \sup_{\pi} \sum_{n=1}^{\infty} |x^*(x_n)|$$

$$= \sum_{n=1}^{\infty} |x^*(x_n)| < \infty$$

★ THEOREM (Vitali-Hahn-Saks-Nikodym): Let (F_n) be a sequence of finitely additive strongly additive vector measures from a σ-field Σ into X. Suppose

$$\lim_n F_n(E) = F(E)$$

exists in norm for all $E \in \Sigma$. Then the sequence (F_n) is uniformly strongly additive. In particular F is strongly additive.

Proof. Notice that

$$\sup_n \|F_n(E)\| < \infty$$

for each $E \in \Sigma$ (convergent seq. are bounded). By Nikodym bounded theorem, (F_n) is a uniform bounded seq. of measures. Suppose (F_n) is not uniformly s.a. Then $\exists$ disjoint seq. (A_m) s.t.

$$\lim_m \|F_n(A_m)\| = 0$$

but not uniformly in n, i.e. $\exists\, n_1 < n_2 < n_3 < \dots$ and disjoint sets $B_1, B_2, \dots$ s.t.

$$\|F_{n_i}(B_i)\| > \delta \quad \forall i$$

for some $\delta > 0$. Choose x_i^* in unit ball of X^* s.t.

$$(*) \qquad x_i^*(F_{n_i}(B_i)) > \delta \quad \forall i$$

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F is strongly additive. Let (E_n) be disjoint. Let $\varepsilon > 0$.
Since (F_n) is u.s.a. $\exists\, n_0$ s.t.

$$\sup_m \| F_m(E_n) \| < \varepsilon \quad \forall n > n_0$$

Let $n > n_0$ and choose m s.t. $\| F(E_n) - F_m(E_n) \| < \varepsilon$. Then

$$\| F(E_n) \| \leq \| F(E_n) - F_m(E_n) \| + \| F_m(E_n) \| < 2\varepsilon$$

Strongly additive $\Rightarrow$ bounded

For the moment assume $\lim F_n(E) = 0$ $\forall E \in \Sigma$. Define $G : \Sigma \to c_0$ by

$$G(E) = \left(x_i^* F_{n_i}(E) \right)$$

Therefore G is a bounded vector measure. Since $l_\infty \not\hookrightarrow c_0$, G is strongly additive. Therefore definition of norm in c_0 implies $(x_i^* F_{n_i})$ is unif. s.a., which contradicts $(*)$

Release assumption that $F_n(E) \to 0$ $\forall E$. Assume not unif. s.a. WLOG we can find a disjoint seq. (E_n) s.t.

$$\|F_n(E_n)\| \geq \delta$$

Find a further subsequence (denoted still by F_n, E_n) s.t.

$$\| F_n(E_n) \| > \delta \text{ but } \| F_n(E_{n+1}) \| < \delta/2$$

To do this notice $\| F_1(E_1) \| > \delta$. $\lim_m F_1(E_m) = 0$, so choose n_1 s.t.

$$\| F_1(E_{n_1}) \| < \delta/2$$

Then $\| F_{n_1}(E_{n_1}) \| > \delta$. Choose $n_2 > n_1$ s.t. $\| F_{n_1}(E_{n_2}) \| < \delta/2$ ... etc. Set $G_n = F_{n+1} - F_n$. Then

$$\lim_n G_n(E) = 0 \quad \forall E \in \Sigma$$

Therefore G_n is unif. s.a. by the earlier part of the proof. But

$$\|G_n(E_{n+1})\| = \|F_{n+1}(E_{n+1}) - F_n(E_{n+1})\|$$

$$\geq \|F_{n+1}(E_{n+1})\| - \|F_n(E_{n+1})\|$$

$$\geq \delta - \delta/2 \geq \delta/2$$

$\Rightarrow$ G_n is not uniformly s.a. $\lightning$

$\boxtimes$

4/12 VECTOR MEASURES

COROLLARY: Let (F_n) be a sequence of countably additive measures on a σ-field which converge set-wise in norm. Then (F_n) is uniform countably additive

COROLLARY: Let μ be a countably additive positive measure. Let (F_n) be a sequence of μ-continuous countably additive vector measures on a σ-field which converges set-wise. Then (F_n) is uniformly μ-continuous

Proof. family of unif. c.a., each μ-cont. $\Rightarrow$ equi-μ-cont. (= unif. μ-cont)

NOTE: c.a. measures, set-wise weak convergence $\Rightarrow$ limit measure is c.a. but does not imply uniform c.a.

Proof. $F_n(E) \longrightarrow F(E)$ weakly $\Rightarrow x^* F_n(E) \longrightarrow x^* F(E)$ $\forall x^*$
Then Vitali-Hahn-Saks $\Rightarrow x^* F$ is c.a., so Orlicz-Pettis $\Rightarrow F$ is c.a.
For counterexample, let F_n: Borel sets $(-\infty,\infty) \longrightarrow L_2(-\infty,\infty)$ be given by

$$F_n(E) = \chi_{E \cap [n,n+1)}$$

Then $F_n(E) \longrightarrow 0$ weakly $\forall E$

16.4a

Let (F_τ) be a family of uniformly c.a. vector measures, each μ-continuous, and such that $F_\tau(E)$ converges setwise. Then each F_τ is bounded and $\sup_\tau \|F_\tau(E)\| < \infty$ for each E, so (F_τ) is uniformly bounded by Nikodym Boundedness Theorem. Define $F: \Sigma \to L_\infty(T)$ by

$$F(E)(\tau) = F_\tau(E)$$

The unf. c.a. $\Rightarrow$ F is countably additive. If $\mu(E) = 0$, then $F_\tau(E) = 0$ $\forall \tau$, so $F(E) = 0$. By Pettis's Theorem, $F \ll \mu$, i.e.

$$\lim_{\mu(E)\to 0} \|F(E)\|_\infty = 0$$

$$\Rightarrow \lim_{\mu(E)\to 0} \sup_\tau \|F_\tau(E)\| = 0$$

Hence (F_τ) is uniformly μ-continuous.

Let $f \in L_2(-\infty,\infty)$. By Holder $f\chi_{E\cap[n,n+1)} \in L_1$, so $\exists N$ s.t. $\int_{\mathbb{R}\setminus[-N,N]} |f\chi_{E\cap[n,n+1)}|\,d\mu < \varepsilon$

Choose $n > N$. Then $\left|\int f\chi_{E\cap[n,n+1)}\,d\mu\right| < \int |f\chi_{E\cap[n,n+1)}|\,d\mu < \varepsilon$. Hence $F_n(E) \to 0$ weakly

F_n is not unf. c.a. Let $E_k = [k,k+1)$. Then for each m,

$$\sup_n \left\|\sum_{k=m}^\infty F_n(E_k)\right\|_2 = \sup_n \left\|\sum_{k=m}^\infty \chi_{E_k\cap[n,n+1)}\right\|_2$$

$$\geq \left\|\sum_{k=m}^\infty \chi_{E_k\cap[m,m+1)}\right\|_2 = \|\chi_{[m,m+1)}\|_2 = 1$$

Hence $\limsup_{m\to\infty} \sup_n \left\|\sum_{k=m}^\infty F_n(E_k)\right\|_2 = 0$ is impossible

Extension of Vector Measures

<u>CARATHEODARY - HAHN - KLUVANEK EXTENSION THEOREM</u>: Let $\mathcal{F}$ be a field of subsets of Ω. Let $F : \mathcal{F} \longrightarrow X$ be a bounded weakly countably additive vector measure. Then F has a (necessarily unique) countably additive extension to $\sigma(\mathcal{F})$ if and only if

① $\exists$ c.a. finite non-negative μ on $\mathcal{F}$ s.t. $F \ll \mu$

iff ② F is strongly additive

iff ③ $F(\mathcal{F})$ is relatively weakly compact

<u>Note</u>: There exist norm c.a. measures on fields that have no c.a. extension. Let Σ = Borel sets in $[0,1]$ and $F : \Sigma \longrightarrow B(\Sigma)$ by $F(E) = \chi_E$. Then $\sum F(E_n)$ does not converge unless there are only finitely non-empty (E_n)'s. Let $\mathcal{F}$ = Stone representation of Σ. Let $\tau : \Sigma \longrightarrow \mathcal{F}$ be the Stone isomorphism. Define $\bar{F} : \mathcal{F} \longrightarrow B(\Sigma)$ by

$$\bar{F}(\tau(E)) = F(E) = \chi_E$$

Then $\bar{F}$ is norm countably additive on $\mathcal{F}$ by default ($\cup E_n \in \mathcal{F}$, E_n disjoint, $\Rightarrow$ only finitely many non-empty) and has no c.a. extension because it is not s.a.

Proof. Let $\bar{F}$ be the extension. Since $\bar{F}(\sigma(\mathcal{F}))$ is relatively weakly compact, so is $F(\mathcal{F})$.

③ $\Rightarrow$ ② Let (E_n) be a monotone sequence. Then $(F(E_n))$ is a sequence in a weakly compact set and $(x^*F(E_n))$ is convergent $\forall x^* \in X^*$. Therefore $F(E_n)$ is weakly convergent.

Therefore F is s.a. (consequence of Rosenthal's lemma).

② $\Rightarrow$ ① :

> Lemma: Let (F_τ) be a family of countably additive vector measures on a σ-field Σ. Let $\mathcal{F}$ be a field s.t. $\sigma(\mathcal{F}) = \Sigma$. Then (F_τ) is uniformly c.a. on Σ iff (F_τ) is uniformly s.a. on $\mathcal{F}$.
>
> Proof. See book

Since F is strongly additive, $\{x^*F : \|x^*\| \leq 1\}$ is unif. s.a. By the lemma, $\{\overline{x^*F} : \|x^*\| \leq 1\}$, where $\overline{x^*F}$ is the Caratheodory-Hahn extension of x^*F from $\mathcal{F}$ to Σ, is unif. c.a. Therefore $\exists$ non-negative finite measure μ on $\sigma(\mathcal{F}) = \Sigma$ s.t. $\overline{x^*F} \ll \mu$ unif. in $\|x^*\| \leq 1$. Therefore $x^*F \ll \mu$ on $\mathcal{F}$ uniformly in $\|x^*\| \leq 1$. Therefore $F \ll \mu$. Hence ② $\Rightarrow$ ①

① $\Rightarrow$ $\exists$ extension : Take μ on $\mathcal{F}$ and extend it to a c.a. $\bar{\mu}$ on $\sigma(\mathcal{F}) = \Sigma$. Let $\Sigma(\bar{\mu})$ = semi-metric space consisting of members of Σ under the semi-metric

$$\rho(A,B) = \bar{\mu}(A \triangle B) = \bar{\mu}(A \setminus B) + \bar{\mu}(B \setminus A)$$

Notice that if $A, B \in \mathcal{F}$, then

$$F(A) - F(B) = F(A \setminus B) - F(B \setminus A)$$

$\mathcal{F}$ is a dense subset of $\Sigma(\bar{\mu})$. This means that if $F : \mathcal{F} \to X$ is viewed as a function from $\mathcal{F} \subset \Sigma(\bar{\mu})$ to X, then F is uniformly continuous. Since F is densely defined, F has a continuous extension $\overline{F}$ to all of Σ and it is easy to check that $\overline{F}$ is additive and $\overline{F} << \overline{\mu}$ (since $\overline{F}$ is continuous on $\Sigma(\overline{\mu})$) Hence $\overline{F}$ is c.a.

$\blacksquare$

<u>COROLLARY</u>: Let $F : \mathcal{F} \to X$ be finitely additive and bounded.

TFAE

1. $\exists$ f.a. μ s.t. $F \ll \mu$
2. F is strongly additive
3. $F(\mathcal{F})$ is relatively weakly compact

Proof. Let $\tilde{\mathcal{F}}$ = Stone representation of $\mathcal{F}$. Define $\tilde{F} : \tilde{\mathcal{F}} \to X$ by

$$\tilde{F}(\tau(E)) := F(E)$$

where $\tau : \mathcal{F} \to \tilde{\mathcal{F}}$. Then $\tilde{F} : \tilde{\mathcal{F}} \to X$ is norm c.a. by default. Then 1, 2, 3 for F in this theorem $\Rightarrow$ 1, 2, 3 respectively in last theorem and conversely. But 1, 2, 3 are equivalent for $\tilde{F}$ $\therefore$ equivalent for F

$\blacksquare$

Other theorems proved by Stone space argument

<u>COROLLARY</u>: unif. s.a. bounded family $\Rightarrow \exists$ f.a. μ s.t. family is unif. μ-cont.

<u>COROLLARY</u>: Let (F_n) be a seq. of strongly additive finitely additive vector measures on Σ. Suppose $F_n \ll \mu$ for some f.a. μ and all n. Suppose $\lim F_n(E)$ exists $\forall E$. Then $F_n \ll \mu$ unif. in n.

(<u>COROLLARY</u>: + unif. bdd + unif. s.a. + μ-cont $\Rightarrow$ unif. μ-cont)

<u>THEOREM</u>: Let $T : \left\{ \begin{matrix} B(\mathcal{F}) \\ L_\infty(\mu) \end{matrix} \right\} \to X$ be a bounded operator. Let $G(E) := T(\chi_E)$. Then T is weakly compact iff G is s.a.

Proof. T weakly compact $\Rightarrow \{T(\chi_E) : E \in \Sigma\}$ is relatively weakly compact $\Rightarrow G(E)$ rel. weakly compact $\Rightarrow G$ s.a. (see previous proof)

To prove T is weakly compact, it suffices to prove

$$\left\{ T\left(\sum \alpha_i \chi_{E_i} \right) : 0 \le \alpha_i \le 1, \ (E_i) \text{ disjoint} \right\}$$

belongs to $\mathrm{co}(G(\mathcal{F})) \subseteq$ w.compact set. To do this look at

$$T\left(\sum \alpha_i \chi_{E_i} \right) = \sum_{i=1}^n \alpha_i G(E_i)$$

WLOG $0 \le \alpha_1 \le \alpha_2 \le \dots \le \alpha_n \le 1$. Observe

$$\sum_{i=1}^n \alpha_i G(E_i) = \alpha_1 G\left(\bigcup_{i=1}^n E_i\right) + (\alpha_2 - \alpha_1) G\left(\bigcup_{i=2}^n E_i\right)$$

$$+ (\alpha_3 - \alpha_2)\, G\left(\bigcup_{i=3}^n E_i\right) + \dots + (\alpha_n - \alpha_{n-1})\, G(E_n)$$

and this belongs to $\mathrm{co}\, G(\mathcal{F})$ since $\alpha_1 + (\alpha_2 - \alpha_1) + \dots + (\alpha_n - \alpha_{n-1}) \le 1$

$\boxtimes$

COROLLARY: If $\ell_\infty \not\hookrightarrow X$, then any $T : \left\{ \begin{matrix} B(\Sigma) \\ L_\infty(\mu) \end{matrix} \right\} \longrightarrow X$ is weakly compact

Proof. G is s.a.

COROLLARY: If $c_0 \not\hookrightarrow X$, then any $T : B(\mathcal{F}) \longrightarrow X$ is w.c.

COROLLARY: c_0 is not complemented in any $L_\infty(\mu)$ space.

Proof. We'll prove that no separable non-reflexive space is complemented in $L_\infty(\mu)$.

To see this let $P : L_\infty(\mu) \longrightarrow X$ be a projection. Since $\ell_\infty \not\hookrightarrow X$, $G(E) := P(\chi_E)$ is s.a., and therefore P is a w.c. projection $\Rightarrow P(L_\infty(\mu))$ is reflexive ↯

Fact (Pelczynski) $L_1[0,1]$ has no non-reflexive second dual subspaces.

Proof. Let $X \overset{T}{\hookrightarrow} L_1[0,1]$. If $X^{**} \hookrightarrow L_1[0,1]$ then X^* is separable. $T : X \longrightarrow L_1[0,1]$, so $T^* : L_\infty[0,1] \longrightarrow X^*$ is weakly compact (since X^* is separable). Hence T is weakly compact and so X is reflexive.

$\boxtimes$

Actually $X \hookrightarrow L_1$, X non-reflexive $\Rightarrow X^*$ not separable

$X \hookrightarrow L_1[0,1]$ non-reflexive $\Rightarrow X^*$ not RNP (via Stegall Ch. 7)

4/17 VECTOR MEASURES

Weak compactness in $L_1(\mu, X)$

THEOREM (Dunford): Let (Ω, Σ, μ) be a finite measure space. Let X and X^* have RNP. A bounded subset K of $L_1(\mu, X)$ is weakly compact if and only if

① K is uniformly integrable (i.e. $\lim_{\mu(E)\to 0} \int \|f\| d\mu = 0$ unif. in $f \in K$)

② $\forall E \in \Sigma$, the set $K_E = \{ \int_E f \, d\mu : f \in K \}$ is weakly compact in X.

Corollary: $L_1(\mu)$ is weakly seq. complete

Toward proving and understanding this theorem we have

LEMMA: If $K \subset L_1(\mu, X)$ is not uniformly integrable but bounded, then there is a sequence $(f_n) \subset K$ and $p \in \mathbb{R}$ s.t. $\forall (\alpha_n) \in \ell_1$

$$p^{-1} \sum_{n=1}^{\infty} |\alpha_n| \leq \| \sum_{n=1}^{\infty} \alpha_n f_n \|_{L_1(\mu, X)} \leq p \sum_{n=1}^{\infty} |\alpha_n|$$

i.e. K contains a copy of the ℓ_1 unit vector basis.

Bourgain 1978: $K \subset L_1(\mu, X)$ unif. integrable + $\exists (f_n) \subset K$ that is a copy of ℓ_1 unit vector basis + (Ω, Σ, μ) = Radon measure space $\Rightarrow \exists t \in \Omega$ s.t. $(f_n(t))$ is a copy of unit vector basis of ℓ_1

Corollary of Bourgain: $\ell_1 \hookrightarrow L_p(\overset{\text{Radon}}{\mu}, X)$ $(1<p<\infty) \Rightarrow \ell_1 \hookrightarrow X$

Kwapian 1976 $c_0 \hookrightarrow L_p(\mu, X)$ $(1 \leq p < \infty)$ $\Rightarrow$ $c_0 \hookrightarrow X$ ⌟

Proof of lemma: Suppose K is not uniformly integrable. Suppose

$$\lim_{\mu(E)\to 0} \int_E \|f\| d\mu = 0$$

is not uniform in $f \in K$. Then the family of measures $\left\{ \int_{(\cdot)} \|f\| d\mu : f \in K \right\}$ is not uniformly c.a. Then $\exists$ disjoint (E_n) in Σ and a sequence (f_n) in K and a $\delta > 0$ s.t.

$$\int_{E_n} \|f_n\| d\mu > \delta \quad \forall n$$

By Rosenthal's lemma, we can assume WLOG

$$\int_{E_n} \|f_n\| d\mu > \delta \quad ; \quad \int_{\bigcup_{i \neq n} E_i} \|f_n\| d\mu < \delta/2 \quad \forall n$$

Let $\beta = \sup_{f \in K} \|f\|_1$. If $(\alpha_n) \in \ell_1$, then

$$\left\| \sum_{n=1}^{\infty} \alpha_n f_n \right\|_1 \leq \sum_{n=1}^{\infty} |\alpha_n| \|f_n\|_1 \leq \beta \sum_{n=1}^{\infty} |\alpha_n|$$

Also, for $(\alpha_n) \in \ell_1$ we have

$$\left\| \sum_{n=1}^{\infty} \alpha_n f_n \right\|_1 = \int_{\Omega} \left\| \sum \alpha_n f_n \right\|_X d\mu$$

$$\geq \int_{\Omega} \left\| \left(\sum_{n=1}^{\infty} \alpha_n f_n \right) \chi_{\bigcup_{m=1}^{\infty} E_m} \right\| d\mu$$

$$\geq \sum_{n=1}^{\infty} \int_{E_n} \|\alpha_n f_n\| \, d\mu - \left\| \sum_{n=1}^{\infty} \alpha_n \cdot f_n \, \chi_{\bigcup_{j \neq n} E_j} \right\|_1$$

Since

$$\left\| \sum_{n=1}^{\infty} \alpha_n f_n \chi_{\bigcup_{j=1}^{\infty} E_j} \right\|_1 = \left\| \sum_{n=1}^{\infty} \alpha_n f_n \chi_{E_n} + \sum_{n=1}^{\infty} \alpha_n f_n \chi_{\bigcup_{j \neq n} E_j} \right\|_1$$

$$\geq \left\| \sum_{n=1}^{\infty} \alpha_n f_n \chi_{E_n} \right\|_1 - \left\| \sum_{n=1}^{\infty} \alpha_n f_n \chi_{\bigcup_{j \neq n} E_j} \right\|_1$$

$$= \sum_{n=1}^{\infty} \|\alpha_n f_n \chi_{E_n}\|_1 - \left\| \sum_{n=1}^{\infty} \alpha_n f_n \chi_{\bigcup_{j \neq n} E_j} \right\|_1$$

Therefore

$$\left\| \sum_{n=1}^{\infty} \alpha_n f_n \right\|_1 \geq \sum_{n=1}^{\infty} \|\alpha_n f_n \chi_{E_n}\|_1 - \sum_{n=1}^{\infty} |\alpha_n| \, \|f_n \chi_{\bigcup_{j \neq n} E_j}\|_1$$

$$= \sum_{n=1}^{\infty} \int_{E_n} \|\alpha_n f_n\| \, d\mu - \sum_{n=1}^{\infty} |\alpha_n| \int_{\bigcup_{j \neq n} E_j} \|f_n\|_X \, d\mu$$

$$\geq \sum_{n=1}^{\infty} |\alpha_n| \int_{E_n} \|f_n\|_X \, d\mu - \sum_{n=1}^{\infty} |\alpha_n| \delta/2$$

$$\geq \sum_{n=1}^{\infty} |\alpha_n| \delta - \sum_{n=1}^{\infty} |\alpha_n| \delta/2 = \frac{\delta}{2} \sum_{n=1}^{\infty} |\alpha_n|$$

$\boxtimes$

This fact proves: K bounded, $K \subset L_1(\mu, X)$, K not unif. integrable $\Rightarrow$ K is not relatively weakly compact (since the ℓ_1 vector basis has no weakly Cauchy subsequence) Hence ① $\Leftarrow$ rel. weak compactness in Dunford's Theorem (with no hypothesis on X)

[Under the conditions K would contain an isomorphic copy of the unit vector basis of ℓ_1, a sequence which is not relatively weakly compact because if it were, the unit ball of ℓ_1 ($=$ abs $\overline{co}$ (unit vector basis)) would be weakly compact and hence ℓ_1 would be reflexive.]

Dunford's original proof of w.c. $\Rightarrow$ unif. int. Suppose $K \subset L_1(\mu, X)$ is relatively weakly compact. It suffices to prove that every seq. in K has a unif. int. subseq. Take a sequence $(f_n) \subset K$. Pass to a weakly convergent subsequence, still called (f_n). If (f_n) is not uniformly int., WLOG then $\exists$ disjoint seq. (E_n) in Σ s.t.

$$\int_{E_n} \|f_n\| \, d\mu > \delta \quad \forall n$$

Even though we are not assuming $L_\infty(\mu, X^*) = L_1(\mu, X)^*$, we can still find a $g_n \in L_\infty(\mu, X^*)$ with $\|g_n\|_\infty = 1$, $g_n = g_n \chi_{E_n}$ s.t.

$$\int_\Omega \langle f_n, g_n \rangle \, d\mu > \delta$$

Set $g = \sum_{n=1}^\infty g_n \chi_{E_n}$. Then $g \in L_\infty(\mu, X^*)$, and

$$(*) \qquad \int_{E_n} \langle f_n, g \rangle \, d\mu > \delta$$

But $f_n \to f \in L_1(\mu, X)$ weakly and $\int_E \langle \cdot, g \rangle \, d\mu \in L_1(\mu, X)^*$ $\forall E \in \Sigma$, so

$$\int_E \langle f_n, g \rangle \, d\mu \longrightarrow \int_E \langle f, g \rangle \, d\mu \quad \forall E \in \Sigma$$

By Vitali-Hahn-Saks, the measures $\int_{(\cdot)} \langle f_n, g \rangle \, d\mu$ are unif. c.a., which contradicts $(*)$

Notice that this says that a weakly convergent sequence in $L_1(\mu)$ has unif. c.a. indefinite integrals. Let μ be defined on $\mathcal{P}(\mathbb{N})$ by

$$\mu(E) = \sum_{n \in E} \frac{1}{2^n}$$

Then ℓ_1 and $L_1(\mu)$ are isometric

$$(\alpha_n) \longleftrightarrow (\alpha_n 2^n)$$

The unif. c.a. of indefinite integrals for functions in K corresponds to

$$\lim_{p\to\infty} \sum_{n=p}^{\infty} |\alpha_n| = 0$$

unif. in $(\alpha_n) \in K \subset \ell_1$. Hence weak convergence $\Rightarrow$ norm convergence in ℓ_1.

HW/ X has Schur property $\Rightarrow \ell_1(X)$ has Schur property

Proof of Dunford's Theorem: $\Rightarrow$ ① already done.
$\Rightarrow$ ②. Fix $E \in \Sigma$. The operator $T_E : L_1(\mu, X) \to X$ given by

$$T_E(f) = \int_E f \, d\mu$$

is continuous. Thus T_E maps weakly compact sets into weakly compact sets. Hence $T_E(K)$ is rel. weakly compact.

② + ① $\Rightarrow$ K rel. w. comp. Take $(f_n) \subset K$. Find a countable field $\mathcal{F} \subset \Sigma$ s.t. each f_n is measurable relative to $\sigma(\mathcal{F})$. List $\mathcal{F} = \{E_i\}$. Use a diagonal procedure + ② to extract a subsequence (g_n) of (f_n) s.t.

$$\lim_n \int_{E_i} g_n \, d\mu =: F(E_i)$$

M6a

$$T : \ell_1 \longrightarrow L_1(\mu)$$

$$T((\alpha_n))(k) := \alpha_k 2^k$$

Then $\| T((\alpha_n)) \|_{L_1} = \sum_{k=1}^{\infty} \int_{\{k\}} |T((\alpha_n))| \, d\mu = \sum_{k=1}^{\infty} |\alpha_k| 2^k \mu(\{k\}) = \sum_{k=1}^{\infty} |\alpha_k|$

$= \|(\alpha_n)\|_{\ell_1}$

Suppose $\alpha^n \to 0$ weakly in ℓ_1. Let ε and choose p s.t. $\sum_{k=p+1}^{\infty} |\alpha_k^n| < \varepsilon/2 \ \forall n$
Let

$$|\beta_k^i| = 0 \quad k > p$$

$$B = \{ \beta^i \in \ell_\infty : |\beta_k^i| = 1 \quad 1 \le k \le p \ , \quad 1 \le i \le 2^p \}$$

For each $i \le 2^p$, $\beta^i(\alpha^n) \to 0$, so $\exists\, n_i$ s.t. $|\beta^i(\alpha^n)| < \varepsilon/2 \ \forall n > n_i$
Let $n_0 = \max\{ n_i : 1 \le i \le 2^p \}$. If $n > n_0$, choose the $\beta^i \in B$ s.t.

$$\beta_k^i = \operatorname{sgn}(\alpha_k^n) \quad 1 \le k \le p$$

Then $\varepsilon/2 \ge |\beta^i(\alpha^n)| = \sum_{k=1}^{p} |\alpha_k^n|$, and so

$$\|\alpha^n\| = \sum_{k=1}^{\infty} |\alpha_k^n| = \sum_{k=1}^{p} |\alpha_k^n| + \sum_{k=p+1}^{\infty} |\alpha_k^n| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Then $\|\alpha^n\| \to 0$.

exists weakly $\forall i$. Now $\mathcal{F}$ is dense in $\sigma(\mathcal{F})$ in the "μ metric"
Therefore

$$\text{w-}\lim_n \int_E g_n \, d\mu = F(E)$$

exists $\forall E \in \sigma(\mathcal{F})$ by ①

($\int_E g_n d\mu$ can be unif. approx. in norm by $\int_A g_n d\mu$ for some $A \in \mathcal{F}$

$E \in \sigma(\mathcal{F}) \Rightarrow \int_E g_n d\mu$ is weakly Cauchy. By ②, $\left(\int_E g_n d\mu\right)$

is weakly convergent. Let $x^* \in X^*$, $\|x^*\| \le 1$. Let $\varepsilon > 0$ and by ①
choose $A \in \mathcal{F}$ s.t.

$$\left\| \int_A g_n d\mu - \int_E g_n d\mu \right\| < \varepsilon/2$$

for all n. Choose p s.t. $m, n \ge p \Rightarrow$

$$\left| x^* \int_A g_n d\mu - x^* \int_A g_m d\mu \right| < \varepsilon/4$$

Then $m, n \ge p \Rightarrow$

$$\left| x^* \int_E g_n d\mu - x^* \int_E g_m d\mu \right| < 8\varepsilon$$

Therefore $\lim \int_E g \, d\mu =: F(E)$ exists weakly $\forall E \in \sigma(\mathcal{F})$.)

Another application of ① $\Rightarrow F \ll \mu$. To see that

$|F|(\Omega) < \infty$, let π be a partition

$$\sum_{E\in\pi} \|F(E)\| \le \sum_{E\in\pi} \varliminf_n \left\| \int_E g_n \, d\mu \right\|$$

$$\le \varliminf_n \sum_{E\in\pi} \left\| \int_E g_n \, d\mu \right\|$$

$$\le \varliminf_n \sum_{E\in\pi} \int_E \|g_n\| \, d\mu$$

$$= \varliminf_n \|g_n\|_1 \le \sup_{f\in K} \|f\|_1 < \infty$$

Hence F is of bdd variation. Since X has RNP, $\exists f \in L_1(\mu, X)$ s.t.

$$F(E) = \int_E f \, d\mu \quad \forall E \in \sigma(\mathcal{F})$$

(f measurable rel. to $\sigma(\mathcal{F})$).

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($\Leftarrow$) Take (f_n) from K. Let $\mathcal{F}$ be a countable field s.t. each (f_n) is measurable relative to $\sigma(\mathcal{F})$. WLOG show (f_n) has a subsequence that converges weakly in $L_1(\sigma(\mathcal{F}), \mu, X)$. By Cantor diagonalization find a subsequence (g_m) of (f_n) s.t.

$$\lim_n \int_E g_n \, d\mu = F(E) \text{ weakly}$$

exists $\forall E \in \mathcal{F}$. Invoke ① to see

$$\lim_n \int_E g_n \, d\mu = F(E) \text{ weakly}$$

exists $\forall E \in \mathcal{F}$. Also by ①, $F \ll \mu$. Additionally,

$$|F|(\Omega) \leq \text{bdd for } \|K\|$$

Since X has RNP, $\exists f \in L_1(\sigma(\mathcal{F}), X)$ s.t.

$$F(E) = \int_E f \, d\mu \quad \forall E \in \sigma(\mathcal{F})$$

Claim: $g_n \to f$ weakly

We know

$$\int_E g_n \, d\mu \to \int_E f \, d\mu \quad \forall E \in \sigma(\mathcal{F})$$

$$\Rightarrow \int_\Omega \langle g_n, h\rangle \, d\mu \longrightarrow \int_\Omega \langle f, h\rangle \, d\mu$$

for all simple functions h in $L_\infty(\sigma(\mathcal{F}), \mu, X^*)$

$$\Rightarrow \int_\Omega \langle g_n, h\rangle \, d\mu \longrightarrow \int_\Omega \langle f, h\rangle \, d\mu$$

for all $h \in K_\infty(\mu, X^*)$

<u>Prelim</u> [X weakly seq. complete , X^* separable $\Rightarrow$ X reflexive]

Let $h \in L_\infty(\mu, X^*)$ be arbitrary. Then by Egoroff we can write

$$h = h\chi_E + h\chi_{\Omega\setminus E}$$

where $h\chi_E \in K_\infty(\mu, X^*)$ and $\mu(\Omega\setminus E)$ is so small that it works in ① for below result

Now

$$\int_\Omega \langle g_n, h\rangle d\mu - \int_\Omega \langle f, h\rangle d\mu = \int_E \big(\langle g_n, h\rangle - \langle f, h\rangle\big) d\mu$$

$$+ \int_{\Omega\setminus E} \langle g_n, h\rangle - \int_{\Omega\setminus E} \langle f - h\rangle \, d\mu$$

By ① this last term is small in norm when $\mu(\Omega \setminus E)$ is small and the first term converges to 0.

Since X^* has RNP, $L_1(\mu, X)^* = L_\infty(\mu, X^*)$, so we have just proved $g_n \to f$ weakly.

$\boxtimes$

Example: If X lacks RNP, then $\exists$ a bounded set $K \subset L_1(\mu, X)$ that obeys ① and ② but s.t. K is not relatively weakly compact

Proof. Take a measure $G : \Sigma \to X$ s.t. $|G|(E) \leq \mu(E)$ but G has no R-N derivative.

Put

$$K = \left\{ g_\pi = \sum_{E \in \pi} \frac{G(E)}{\mu(E)} \chi_E : \pi \text{ a partition of } \Omega \right\}$$

Then K is bounded ($\|g_\pi\|_X \leq 1$ a.e.), so K is uniformly integrable. Take $A \in \Sigma$, and consider

$$\int_A g_\pi \, d\mu = \sum_{E \in \pi} \frac{G(E)}{\mu(E)} \mu(E \cap A) \in \overline{co}\, G(\Sigma) = \text{w.c.}$$

summation by parts

Hence K satisfies ①, ②

Suppose K is relatively w.c. in $L_1(\mu, X)$. Then (g_π) is a net in a weakly compact set, so it has a weakly convergent subnet (g_β) which must converge to a derivative of G ↯.

Example ① + ② fail to characterize w. compactness in $L_1(\mu, \ell_1)$ ← Lebesgue measure on $[0,1]$

Yet

$$f_n(t) = (0, 0, \ldots, 0, r_n(t), 0, \ldots)$$

← radamacher

Notice that

$$\forall E \in \Sigma \quad \int_E f_n \, d\mu = \left(0, 0, \ldots, 0, \int_E r_n(t)\, d\mu, 0, \ldots\right) \to 0$$

in ℓ_1-norm. Functions of the form $x^* \chi_E \in L_\infty(\mu, X^*)$ separates points of $L_1(\mu, X)$, i.e.

$$\int_E x^* f \, d\mu = \int_E x^* g \, d\mu \quad \forall x^* \in X^*, \forall E \in \Sigma$$

implies $f = g$ a.e.. Hence if (f_n) has a weakly convergent subsequence, then the weak limit of that subsequence must be 0.

Define ℓ on $L_1(\mu, \ell_1)$ by

$$\ell(f) := \sum_{n=1}^{\infty} \int_\Omega r_n \varphi_n \, d\mu \quad \text{for } f = (\varphi_1, \varphi_2 \ldots)$$

Notice

$$|\ell(f)| \le \sum_{n=1}^{\infty} \int_\Omega |\varphi_n| \, d\mu = \|f\|_{L_1}$$

Also notice that

$$\ell(f_n) = \int_\Omega r_n \cdot r_n \, d\mu = 1 \quad \lightning$$

Hence no weakly convergent subsequence.

C(K) OPERATOR THEORY

Let K be a compact Hausdorff space. Let Σ be the Borel σ-field of subsets of K. Then $C(K)^* =$ all regular Borel measures

$$\ell(f) = \int f \, d\mu \qquad \|\ell\| = |\mu|$$

Notice $B(\Sigma)$ sits in $C(K)^{**}$ as a closed subspace. For if $\mu \in C(K)^*$ and $f \in B(\Sigma)$, then

$$\ell_f(\mu) := \int_K f \, d\mu$$

puts $B(\Sigma)$ into $C(K)^{**}$ with

$$\| \ell_f(\mu) \| \leq |\mu| \, \|f\|_{B(\Sigma)}$$

Take $T : C(K) \longrightarrow X$ arbitrary bounded linear operator. Then $T^{**} : C(K)^{**} \longrightarrow X^{**}$.

$$C(K) \subseteq B(\Sigma) \subseteq C(K)^{**}$$

Consider $\hat{T} : B(\Sigma) \to X^{**}$ given by $T^{**}|_{B(\Sigma)}$. Let G be the representing measure for $\hat{T}$

$$\hat{T}(f) = \int f\,dG$$

$$G(E) = \hat{T}(\chi_E)$$

$$T(f) = \int f\,dG$$

We call G the representing measure for T.

Notice that T weakly compact $\Rightarrow T^{**}$ weakly compact $\Rightarrow \hat{T}$ weakly compact $\Rightarrow T$ weakly compact

$\uparrow$ $\hat{T}$ extends T

i.e. T is weakly compact $\Leftrightarrow \hat{T}$ is weakly compact.

<u>COROLLARY</u>: $l_\infty \not\hookrightarrow X^{**} \Rightarrow$ all $T : C(K) \to X$ are weakly compact.

Proof. $l_\infty \not\hookrightarrow X^{**} \Rightarrow$ all $\hat{T} : B(\Sigma) \to X^{**}$ are weakly compact $\Rightarrow$ all $T : C(K) \to X$ are weakly compact

<u>COROLLARY</u>: T is weakly compact $\Leftrightarrow G$ is strongly additive

Proof. $\hat{T}$ is weakly compact $\Leftrightarrow G$ is strongly additive

Consider $x^* \in X^*$ and look at

$$x^*G(E) = x^* \hat{T}(\chi_E) = x^* T^{**}(\chi_E) = T^*(x^*)(\chi_E)$$

$$= \mu_{T^*(x^*)}(E)$$

Therefore x^*G is regular $\forall x^* \in X^*$, so G is weak*-countably additive

What does this mean? T weakly compact $\Leftrightarrow T^{**}$ has its range in X

$\Rightarrow \hat{T}$ has range in $X \Rightarrow G$ is X-valued

$\Rightarrow G$ is weakly c.a. $\Rightarrow G$ is c.a. $\Rightarrow G$ is s.a.

$\Rightarrow \hat{T}$ is weakly compact $\Rightarrow T$ weakly compact

<u>THEOREM</u> (Bartle-Dunford-Schwartz) $T : C(K) \to X$ is weakly compact

$\Leftrightarrow G$ has its values in X

$\Leftrightarrow G$ is c.a.

$\Leftrightarrow G$ is s.a.

Can show: $B(C(K), X) \cong$ space of all f.a. vector measures G from Σ (Borel) to X^{**} s.t. x^*G is regular $\forall x^*$

$$T(f) = \int f\, dG$$

$$\|T\| = \|G\|(K)$$

DEFINITION: K is Stonean if the closure of every open set is open.

Fact: $C(K)$ is order complete $\Leftrightarrow$ K is Stonean

i.e. if (f_α) is a net in $C(K)$ s.t. $f_\alpha \downarrow$ as $\alpha \uparrow$ and $\exists\, g \in C(K)$ s.t. $f_\alpha \geq g$ $\forall \alpha$, then the set (f_α) has a g.l.b. in $C(K)$

Suppose Y is a closed subspace of W. Let $T : Y \to C(K)$ be a bounded linear operator. If T has a one-dimensional range, the Hahn-Banach theorem produces an operator $S : W \to C(K)$ s.t. S agrees with T on Y. Notice $\|T\|$ = value of the smallest constant function φ in $C(K)$ s.t. $|Ty| \leq \varphi$ for all y in the unit ball of Y. You can use this to mimic the usual proof of H-B to produce $S : W \to C(K)$ that agrees with T on $C(K)$ (IF $C(K)$ is order complete - removing assumption that range T is 1-dimensional)

Application: Suppose $C(K)$ is order complete and is a closed subspace of another Banach space Z. Then

$$\begin{tikzcd} Z \arrow[dr, dashed, "S"] & \\ C(K) \arrow[u, "J"] \arrow[r, "I"'] & C(K) \end{tikzcd}$$

there exists $S : Z \to C(K)$ s.t. S agrees with I on $C(K)$. Therefore S is a projection. Therefore $C(K)$ is complemented in any Banach space in which it resides (if $C(K)$ is order complete) In particular, $C(K)$ is complemented in $B(\Sigma)$.

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<u>COROLLARY</u>: Order complete $C(K)$'s are norm-one complemented in any B-space in which they reside

<u>COROLLARY</u>: Let $C(K)$ be order complete. If $\ell_\infty \not\hookrightarrow X$, then every bounded linear operator $T : C(K) \longrightarrow X$ is weakly compact

Proof. $C(K)$ is a subspace of $B(\Sigma)$

$$\begin{array}{ccc} B(\Sigma) & & \\ {\scriptstyle P}\downarrow & \overset{S = TP}{\searrow} & \\ C(K) & \xrightarrow[T]{} & X \end{array}$$

But since $\ell_\infty \not\hookrightarrow X$, S is weakly compact, so T is weakly compact because $T(f) = S(f)$ $\forall f \in C(K)$ and

$$T(B_{C(K)}) \subset S(B_{B(\Sigma)}) \subset W.\text{compact set}$$

$\blacksquare$

<u>THEOREM</u> (Bartle-Dunford-Schwartz-Grothendieck) Let K be any compact Hausdorff space. A weakly compact $T : C(K) \longrightarrow X$ maps weak Cauchy sequences into norm convergent sequences.

Consequently weakly compact operators carry weakly compact sets into norm compact sets, i.e. $C(K)$ has the Dunford-Pettis property.

Proof. Let (f_n) in $C(K)$ be weakly Cauchy, i.e. there exists a Baire-1 function f s.t. $f_n \to f$ pointwise, with $f \in B(\Sigma)$.

$$T(f_n) = \int_K f_n dG$$

[Recall w. Cauchy $\Rightarrow \exists\, p$ s.t. $|f_n(t)| \leq p \;\; \forall n \; \forall t$

$$\hat{T}(f) = \int_K f\, dG$$

G representing measure. Take a c.a. measure μ s.t. $G \ll \mu$. Then $f_n \to f$ μ almost uniformly. Let E be a set of uniform convergence s.t. $\mu(K \setminus E)$ is so small that $\|G\|(K\setminus E) < \varepsilon/2$

$$\|T(f_n) - \hat{T}(f)\| \leq \underbrace{\left\| \int_E (f_n - f)\, dG \right\|}_{A_n} + \underbrace{\left\| \int_{\Omega \setminus E} (f_n - f)\, dG \right\|}_{B_n}$$

Then

$$A_n \leq \sup_{t \in E} |f_n(t) - f(t)| \, \|G\|(E) \leq \varepsilon/2 \quad \text{for } n > n_0$$

$$B_n \leq \sup_{t \in \Omega\setminus E} |f_n(t) - f(t)| \, \|G\|(\Omega \setminus E) \leq p\,\varepsilon/2$$

Hence $\|Tf_n - \hat{T} f\| \to 0$

$\blacksquare$

<u>**COROLLARY**</u>: Any reflexive complemented subspace of $B(\Sigma)$, $L_\infty(\mu)$, or ~~order complete~~ $C(K)$ is finite dimensional.

Proof. Notice that all of these spaces are $C(K)$ spaces (Kakutani Representation) Let R be a complemented subspace of one of them. Let $P: C(K) \to R$ be the projection. Then $P(B_{C(K)})$ is weakly compact, and so $P(P(B_{C(K)}))$ is compact. Hence R has a relatively compact open subset, so R is finite dimensional

$\boxtimes$

<u>COROLLARY</u>: Any complemented infinite dimensional subspace of $B(\Sigma)$, $L_\infty(\mu)$ or order-complete $C(K)$ contains a copy of ℓ_∞.

order complete

Proof. If X is a complemented subspace of $C(K)$ and $P: C(K) \to X$ is a projection, then $\ell_\infty \not\hookrightarrow X$ implies P is weakly compact. Hence X is reflexive $\Rightarrow$ X is finite dimensional.

Goal: If K is an arbitrary compact T_2 space and $T: C(K) \to X$ is not weakly compact, then $\exists\, Y \subset C(K)$ s.t. Y is isometric to c_0 and $T|_Y$ is an isomorphism.

<u>LEMMA</u>: Let Σ be the σ-field of Borel sets in a compact Hausdorff space K. Let $\mathcal{M}$ be a family of regular measures on Σ. Then TFAE

① $\forall$ disjoint sequences (O_n) of open sets, $\lim \mu(O_n) = 0$ unif. in $\mu \in \mathcal{M}$

② $\forall$ disjoint sequences (O_n) of open sets $\lim |\mu|(O_n) = 0$ unif in $\mu \in \mathcal{M}$

③ $\mathcal{M}$ is uniformly inner regular on the open sets, i.e. for $\forall \varepsilon > 0$ and each open O there exists compact $K \subseteq O$ s.t. $\sup |\mu|(O \setminus K) < \varepsilon$

④ $\mathcal{M}$ is uniformly inner regular

⑤ $\mathcal{M}$ is uniformly countably additive

⑥ $\mathcal{M}$ is uniformly regular, i.e. for each Borel set E and $\varepsilon > 0$ there exist open O and compact K with $O \supseteq E \supseteq K$ and

$$\sup_{\mu \in \mathcal{M}} |\mu|(O \setminus K) < \varepsilon$$

HW/ ① Let $rca(\Sigma)$ = all regular Borel measures on Borel sets of a compact Hausdorff space. Let (μ_n) be a bdd seq. in this space. Then $\mu_n \to 0$ weakly iff $\mu_n(O) \to 0$ for all open sets O and

$$\lim_m \mu_n(O_m) = 0$$

unif. in n $\forall$ disjoint seq. (O_m) of open sets.

② Let $G : \Sigma \to X$ be a weakly c.a. vector measure s.t. $x^* G$ is regular $\forall x^* \in X^*$. Prove G is c.a. and G is norm regular in the sense that for each $\varepsilon > 0$ and Borel set E, there exists open O and compact K s.t. $O \supseteq E \supseteq K$ and $\|G\|(O \setminus K) < \varepsilon$

Proof of lemma: ① $\Rightarrow$ ② Suppose (O_n) is a disjoint sequence of open sets and $\exists$ a sequence (μ_n) in $\mathcal{M}$ s.t.

$$\inf |\mu_n|(O_n) > \delta$$

Then $O_n = E_n^+ \cup E_n^-$, so $\forall n$ there exist Borel $F_n \subset O_n$ s.t.

$$|\mu_n(F_n)| > \delta/2$$

Use regularity to find open G_n s.t. $O_n \supset G_n \supset F_n$ s.t.

$$|\mu_n(G_n)| > \delta/2$$

Then (G_n) is a disjoint seq. of open sets which contradicts ①

② $\Rightarrow$ ③ Let O be open and $\varepsilon > 0$. Suppose ③ fails, i.e. $\nexists$ no compact $K \subset O$ s.t.

$$\sup_{\mu \in \mathcal{M}} |\mu|(O \setminus K) \leq \varepsilon$$

Then $\exists \mu_1 \in \mathcal{M}$ s.t. $\mu_1(O) > \varepsilon$. Since $|\mu_1|$ is regular, $\exists$ compact $K_1 \subset O$ s.t.

$$|\mu_1|(K_1) > \varepsilon$$

Since K is normal, $\exists$ open O_1 s.t. $O \supseteq \overline{O}_1 \supseteq O_1 \supseteq K_1$. Then

$$|\mu_1|(\overline{O}_1) \geq |\mu_1|(O_1) \geq |\mu_1|(K_1) > \varepsilon$$

Now $\overline{O}_1$ is a compact subset of O, so $\exists \mu_2 \in \mathcal{M}$ with

$$|\mu_2|(O \setminus \overline{O}_1) > \varepsilon$$

Regularity of $|\mu_2|$ implies $\exists$ compact K_2 s.t. $\mathcal{O} \supset K_2 \supset \overline{\mathcal{O}}_1$ and

$$|\mu_2|(K_2 \setminus \overline{\mathcal{O}}_1) > \varepsilon$$

By normality $\exists$ open $\mathcal{O}_2$ s.t.

$$\mathcal{O} \supset \overline{\mathcal{O}}_2 \supseteq \mathcal{O}_2 \supseteq K_2 \supseteq \overline{\mathcal{O}}_1 \supseteq \mathcal{O}_1$$

Observe

$$|\mu_2|(\mathcal{O}_2 \setminus \overline{\mathcal{O}}_1) \geq |\mu_2|(K_2 \setminus \overline{\mathcal{O}}_1) > \varepsilon$$

$\exists$ $\mu_3 \in \mathcal{M}$ s.t. $|\mu_3|(\mathcal{O} \setminus \overline{\mathcal{O}}_2) > \varepsilon$, so $\exists$ compact K_3 with $\mathcal{O} \supseteq K_3 \supseteq \overline{\mathcal{O}}_2$ and

$$|\mu_3|(K_3 \setminus \overline{\mathcal{O}}_2) > \varepsilon$$

Find $\mathcal{O}_3$ s.t.

$$\mathcal{O} \supseteq \overline{\mathcal{O}}_3 \supseteq \mathcal{O}_3 \supseteq K_3 \supseteq \overline{\mathcal{O}}_2 \supseteq \mathcal{O}_2$$

Get

$$|\mu_3|(\mathcal{O}_3 \setminus \overline{\mathcal{O}}_2) \geq |\mu_3|(K_3 \setminus \mathcal{O}_2) > \varepsilon$$

Continue this to produce $\mathcal{O}_n \uparrow$, $\mathcal{O}_n$ open,

$$\mathcal{O} \supseteq \overline{\mathcal{O}}_{n+1} \supseteq \mathcal{O}_{n+1} \supseteq \overline{\mathcal{O}}_n \supseteq \mathcal{O}_n$$

and produce measures μ_n in $\mathcal{M}$ such that

$$|\mu_{n+1}|(\mathcal{O}_{n+1} \setminus \overline{\mathcal{O}}_n) > \varepsilon$$

Put $G_{n+1} = \mathcal{O}_{n+1} \setminus \overline{\mathcal{O}}_n$ (open). Then (G_n) is a disjoint seq. of open sets and $|\mu_{n+1}|(G_{n+1}) > \varepsilon$ $\forall n$. This contradicts ②.

③ $\Rightarrow$ ④ Call a Borel set E $\mathcal{M}$-measurable if there exists for each ε a compact K_E s.t.

$$\text{a)} \quad \sup_{\mu \in \mathcal{M}} |\mu|(K \setminus K_E) < \varepsilon$$

$$\text{b)} \quad K_E \cap E \text{ compact}$$

Observe

$$|\mu|(E \setminus K_E) \leq |\mu|(K \setminus K_E) < \varepsilon$$

If we can prove every Borel set is $\mathcal{M}$-measurable, then we shall have proved ④

Observe every compact set is $\mathcal{M}$ measurable (Take $K_E = K$)
Observe every open set is $\mathcal{M}$-measurable (Take K_0' from ③ s.t.

$$\sup |\mu|(\mathcal{O} \setminus K_0') < \varepsilon$$

and put $K_0 = K_0' \cup K \setminus O$.

Claim: The M-measurable sets are closed under countable intersection.

Let (E_n) be a seq. of M-measurable sets. Then $\exists$ seq. (K_n) compact s.t.

$$\sup_{\mu \in M} |\mu|(K \setminus K_n) < \varepsilon/2^n$$

and $E_n \cap K_n$ compact $\forall n$. Now

$$\left(\bigcap_{n=1}^{\infty} E_n\right) \cap \left(\bigcap_{n=1}^{\infty} K_n\right) = \bigcap_{n=1}^{\infty} (E_n \cap K_n)$$

is compact, and

$$\sup_{\mu \in M} |\mu| \left(K \setminus \bigcap_{n=1}^{\infty} K_n\right)$$

$$\leq \sum_{n=1}^{\infty} \sup_{\mu \in M} |\mu|(K \setminus K_n) < \varepsilon$$

Claim: The M-measurable sets are closed under complementation

Let E be M-measurable, $\varepsilon > 0$. $\exists K_1$ compact st $E \cap K_1$ is compact and

$$\sup_{\mu \in \mathcal{M}} |\mu|(K \setminus K_1) < \varepsilon/2$$

Now $K \setminus (E \cap K_1)$ is open, and hence it is $\mathcal{M}$-measurable. Therefore $\exists$ compact K_2 s.t. $K_2 \cap (K \setminus (E \cap K_1))$ is compact and

$$\sup_{\mu \in \mathcal{M}} |\mu|(K \setminus K_2) < \varepsilon/2$$

Then $K_1 \cap K_2 \cap K \setminus E = K_1 \cap K_2 \cap (K \setminus (E \cap K_1))$ is compact and

$$\sup_{\mu \in \mathcal{M}} |\mu| (K \setminus (K_1 \cap K_2)) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Therefore $K \setminus E$ is $\mathcal{M}$-measurable

Hence $\mathcal{M}$-measurable is a σ-field including the open sets, so every Borel set is $\mathcal{M}$-measurable.

4/26 VECTOR MEASURES

④ $\Rightarrow$ ⑤ Enough to prove (E_n) decreasing sequence of Borel sets s.t. $E_n \downarrow \phi \Rightarrow$

$$\sup_{\mu \in \mathcal{M}} |\mu(E_n)| \to 0$$

Let $\varepsilon > 0$, and use ④ to find compact sets $K_n \subseteq E_n$ and

$$\sup_{\mu \in \mathcal{M}} |\mu|(E_n \setminus K_n) < \varepsilon/2^n$$

But $\bigcap_{n=1}^{\infty} K_n = \phi$, so $\exists m_0$ such that $\bigcap_{n=1}^{m_0} K_n = \phi$. Let $\underline{m \geq m_0}$ and compute

$$\sup_{\mu \in \mathcal{M}} |\mu|(E_m) = \sup_{\mu \in \mathcal{M}} |\mu|\left(E_m \setminus \bigcap_{n=1}^{m} K_n\right)$$

$$= \sup_{\mu \in \mathcal{M}} |\mu|\left(\bigcup_{n=1}^{m} E_m \setminus K_n\right)$$

$$\leq \sup_{\mu \in \mathcal{M}} \sum_{n=1}^{m} |\mu|(E_m \setminus K_n) \leq \varepsilon$$

⑤ $\Rightarrow$ ⑥ Let (μ_n) be a sequence in $\mathcal{M}$ and take a Borel set E. Take a sequence (O_n) of open sets and a sequence (K_n) of compact sets s.t.

$$K_n \subset E \subset O_n$$

$$|\mu_n|(O_n \setminus K_n) < \varepsilon \quad \forall n$$

Put $G = \bigcap_{n=1}^{\infty} O_n$ and $F = \bigcup_{n=1}^{\infty} K_n$. Then

$$|\mu_n|(G \setminus F) < \varepsilon \quad \forall n$$

Since $\mathcal{M}$ is uniformly c.a.

$$|\mu_n|\left(\bigcap_{k=1}^{m} O_k \setminus \bigcup_{j=1}^{m} K_j \right) \xrightarrow[m\to\infty]{} |\mu_n|(G \setminus F) < \varepsilon$$

uniformly in n. Take m_0 s.t.

$$|\mu_n|\left(\bigcap_{k=1}^{m_0} O_k \setminus \bigcup_{j=1}^{m_0} K_j \right) < \varepsilon \quad \forall n$$

Hence there exists open O and closed K s.t. $K \subset E \subset O$ and

$$\sup |\mu_n| (O \setminus F) < \varepsilon$$

Therefore the sequence (μ_n) is uniformly regular. Since (μ_n) was selected arbitrarily from $\mathcal{M}$, we see $\mathcal{M}$ is uniformly regular.

⑥ $\Longrightarrow$ ⑦. Take a disjoint sequence of open sets (O_n). Want to show $\lim \mu(O_n) = 0$ unif. in $\mu \in \mathcal{M}$. To this end, notice $\bigcup O_n$ is open. Let $\varepsilon > 0$. By unif. regularity, $\exists$ compact $K \subset \bigcup O_n$ s.t.

$$\sup_{\mu \in \mathcal{M}} |\mu| \left(\bigcup_{n=1}^{\infty} O_n \setminus K \right) < \varepsilon$$

By compactness $\exists\, n_0$ s.t. $K \subset \bigcup_{n=1}^{n_0} O_n$. If $m \geq n_0+1$

$$|\mu(O_m)| \leq |\mu| \left(\bigcup_{n=n_0+1}^{\infty} O_n \right) \leq |\mu| \left(\bigcup_{n=1}^{\infty} O_n \setminus K \right) < \varepsilon$$

$\blacksquare$

<u>**COROLLARY**</u>: $T : C(K) \longrightarrow X$ is weakly compact if and only if G is regular

Proof. Let $\mathcal{M} = \{ x^*G : \|x^*\| \leq 1 \}$. This is a family of regular measures which is unif. c.a. if and only if G is c.a. iff $\mathcal{M}$ is unif. regular iff G is regular

$\blacksquare$

<u>**COROLLARY**</u>: $T : C(K) \to X$ is weakly compact if and only if $x^*G(O_n) \to 0$ unif. in $\|x^*\| \leq 1$ for each disjoint sequence (O_n) of open sets.

Proof. Put $\mathcal{M} = \{ x^*G : \|x^*\| \leq 1 \}$. T is weakly compact $\Leftrightarrow$ $\mathcal{M}$ is unif. c.a. $\Leftrightarrow$ $\mathcal{M}$ obeys ① of lemma.

THEOREM: Let $T: C(K) \longrightarrow X$ be a non-weakly compact operator. Then there exists a subspace Y of $C(K)$ that is isometric to c_0 s.t. $T|_Y$ is an isomorphism. Consequently, $c_0 \not\hookrightarrow X$ implies all $T: C(K) \to X$ are weakly compact.

Proof. Suppose $T: C(K) \longrightarrow X$ is not weakly compact. Then $\{x^*G : \|x^*\| \leq 1\}$ is not unif. c.a. By the lemma, there exists a disjoint seq. (O_n) of open sets and a sequence (x_n^*) in the unit ball of X^*, and an $\varepsilon > 0$ s.t.

$$|x_n^* G(O_n)| > \varepsilon$$

Use Rosenthal's lemma to get $|x_n^* G(O_n)| > \varepsilon$ but

$$|x_n^* G| \left(\bigcup_{m \neq n} O_m \right) < \varepsilon/2$$

(relabelling if necessary) Observe that $x_n^* G$ is regular $\forall n$. For each n there exists $f_n \in C(K)$ s.t. f_n vanishes outside O_n and $\|f_n\| = 1$
$0 \leq f_n \leq 1$
and

$$\left| \int_K f_n \, d(x_n^* G) \right| > \varepsilon$$

Set

$$Y = \left\{ \sum_{n=1}^{\infty} \alpha_n f_n : (\alpha_n) \in c_0 \right\}$$

Observe

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let $2\delta < |x_n^* G(O_n)| - \varepsilon$. Choose compact $K_n \subset O_n$ st

$$|x_n^* G|(O_n \setminus K_n) < \delta$$

Then

$$|x_n^* G(O_n \setminus K_n)| < \delta$$

$$\Rightarrow |x_n^* G(O_n) - x_n^* G(K_n)| < \delta$$

$$\Rightarrow |x_n^* G(K_n)| > |x_n^* G(O_n)| - \delta$$

$$\left| \int_K f_n \, d(x_n^* G) \right| = \left| \int_{O_n} f_n \, d(x_n^* G) \right| \geq \left| \int_{K_n} f_n \, d(x_n^* G) \right| - \left| \int_{O_n \setminus K_n} f_n \, d(x_n^* G) \right|$$

$$\geq |x_n^* G(K_n)| - |x_n^* G|(O_n \setminus K_n)$$

$$\geq |x_n^* G(O_n)| - \delta - \delta > \varepsilon$$

$$\left\| \sum \alpha_n f_n \right\|_{C(K)} = \sup |\alpha_n| = \|(\alpha_n)\|_{c_0}$$

Hence Y is isometric to c_0

Let $\sum \alpha_n f_n \in Y$. Consider

$$\left| x_m^* T \left(\sum \alpha_n f_n \right) \right|$$

$$\geq \left| x_m^* T(\alpha_m f_m) \right| - \left| \sum_{n \neq m}^{\infty} \alpha_n x_m^* T(f_n) \right|$$

$$\geq |\alpha_m| \varepsilon - \left| \int_{\bigcup_{n \neq m} O_n} \sum \alpha_n f_n \, d x_m^* G \right|$$

$$\geq |\alpha_m| \varepsilon - \sum_{n \neq m} |\alpha_n| \left| \int_{O_n} f_n \, d x_m^* G \right|$$

$$\geq |\alpha_m| \varepsilon - \sum_{n \neq m} |\alpha_n| \int_{O_n} d|x_m^* G|$$

$$\geq |\alpha_m| \varepsilon - \|(\alpha_n)\|_{c_0} \, |x_m^* G| \left(\bigcup_{n \neq m} O_n \right)$$

$$\geq |\alpha_m| \varepsilon - \|(\alpha_n)\|_{c_0} \, \varepsilon/2$$

Therefore

$$\left\| T \left(\sum \alpha_n f_n \right) \right\| \geq \frac{\varepsilon}{2} \|(\alpha_n)\|_{c_0}$$

Therefore T^{-1} is continuous, so T is an isomorphism on Y.

$\boxtimes$

<u>**Corollary**</u>: $T : C(K) \to X$. TFAE

① T is unconditionally converging, (i.e. T takes WUC's to UC's)

② T is weakly compact

③ T takes weakly null sequences to norm null sequences

④ T takes weakly Cauchy seq. into norm convergent seq.

⑤ If (f_n) is a bounded seq. in $C(K)$ s.t. $f_n f_m = 0$ for $m \neq n$ then $T(f_n) \to 0$.

Proof. ① ⇒ ② If T is not weakly compact, T preserves a copy of c_0. c_0 has plenty of WUC's which are not UC's ($\sum_{k=1}^{\infty} e_k$, e.g.). Hence T is not unconditionally converging

$\boxed{\ast \text{ prel}}$

② ⇒ ① (is true in general) Let $T : Y \to X$ be weakly compact. Let $\sum y_n$ be a WUC. Then the sum of every finite subseries of $\sum y_n$ lies in a fixed ball. Hence the sum of every finite subseries of $\sum Ty_n$ lies in a fixed weakly compact set. But every subseries of $\sum Ty_n$ is weakly Cauchy because $\sum Ty_n$ is also a WUC. Hence every subseries of $\sum Ty_n$ is weakly convergent. Orlicz-Pettis implies every subseries of $\sum Ty_n$ is norm convergent, i.e. $\sum Ty_n$ is an UC

② ⇒ ⑤ If $f_n \not\to 0$ in norm, then (f_n) has a subseq. that is equivalent to the unit vector basis of c_0. The unit vector basis of c_0 tends to zero weakly. Hence T(that subseq) $\to 0$ in norm (Dunford-Pettis), and so every subseq. (f_n) has a subseq. mapped to a null seq. by T. Hence T takes the whole seq. into a null seq.

5 $\Rightarrow$ 2 If T is not weakly compact, the f_n's in the last theorem satisfy $\lim \|T(f_n)\| \neq 0$ ↯

4 $\Rightarrow$ 3

2 $\Rightarrow$ 3 Dunford-Pettis

3 $\Rightarrow$ 4 argument p177

2 $\Rightarrow$ 4

3 $\Rightarrow$ 5 If (f_n) are as in 5, then $f_n \to 0$ weakly. For

$$\left|\int_K f_n \, d\mu\right| = \left|\int_{|f_n|>0} f_n \, d\mu\right| = \left|\int_{\underset{\text{disjoint}}{O_n}} f_n \, d\mu\right|$$

$$\leq \underset{\text{bdd for } (f_n)}{K} \, |\mu|(O_n) \to 0 \quad \text{by c.a. of } |\mu|$$

$\blacksquare$

5/1 Vector Measures

Absolutely summing - Pietsch integral - Nuclear operators

<u>Definition</u>: An operator $T: X \to Y$ is called absolutely summing if it takes WUC into absolutely converging series.

Fact: T is absolutely summing $\Leftrightarrow$ T takes UC's into absolutely convergent series $\Leftrightarrow$ $\exists K$ s.t. for $x_1, \dots, x_n \in X$

$$\sum_{i=1}^{n} \|Tx_i\| \leq K \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^{n} |x^*(x_i)| \right)$$

(consider map from space of WUC's to space of abs. conv. series)

The smallest constant K s.t. above holds is called the absolutely summing norm of T and is written $\|T\|_{as}$

The class of a.s. operators from X to Y is an "ideal" in the following sense: if $R: Z \to X$ and $S: Y \to W$ are bounded linear operators, then $STR: Z \to W$ is absolutely summing and

$$\|STR\|_{as} \leq \|S\| \|T\|_{as} \|R\|$$

provided T is a.s.

THEOREM: $T : C(K) \to X$ is a.s. $\Leftrightarrow$ G is of bounded variation. In this case

$$\|T\|_{as} = |G|(K)$$

Proof. Suppose $|G|(K) < \infty$. Take $(f_1, \dots, f_m)$ in $C(K)$.

$$\sum_{n=1}^{m} \|T(f_n)\| = \sum_{n=1}^{m} \left\| \int_K f_n \, dG \right\| \leq \sum_{n=1}^{m} \int_K |f_n| \, d|G|$$

$$= \int_K \left(\sum_{n=1}^{m} |f_n| \right) d|G|$$

$$\leq |G|(K) \left\| \sum_{n=1}^{m} |f_n| \right\|_{C(K)}$$

Observe

$$\left\| \sum_{n=1}^{m} |f_n| \right\|_{C(K)} = \sup_{|\varepsilon_n|=1} \left\| \sum_{n=1}^{m} \varepsilon_n f_n \right\|_{C(K)}$$

$$= \sup_{\substack{|\varepsilon_n|=1 \\ |\mu|(K) \leq 1 \\ \mu \in C(K)^*}} \int_K \left(\sum_{n=1}^{m} \varepsilon_n f_n \right) d\mu$$

$$= \sup_{\substack{|\varepsilon_n|=1 \\ |\mu|(K) \leq 1}} \sum_{n=1}^{m} \varepsilon_n \int_K f_n \, d\mu$$

$$= \sup_{|\mu|(K)\le 1} \sum_{n=1}^{m} \left| \int_K f_n \, d\mu \right| = \text{WUC norm } (f_n)$$

Hence

$$\sum_{n=1}^{m} \|T(f_n)\| \le |G|(K) \sup_{|\mu|(K)\le 1} \sum_{n=1}^{m} \left| \int_K f_n \, d\mu \right|$$

and so T is a.s. with $\|T\|_{a.s.} \le |G|(K)$.

Now suppose T is a.s. Let $O_1, O_2, \ldots, O_m$ be a finite disjoint family of open sets. If $f_1, \ldots, f_m$ are in $C(K)$ and each f_i is supported on O_i, then ($\|f_i\| \le 1$)

$$\sum_{n=1}^{m} \left| \int_K f_n \, d\mu \right| \le |\mu|(K)$$

for all $\mu \in C(K)^*$. Now T a.s. implies

$$\sum_{n=1}^{m} \|T(f_n)\| \le \|T\|_{a.s.} \sup_{|\mu|(K)\le 1} \sum_{n=1}^{m} \left| \int_K f_n \, d\mu \right|$$

$$\le \|T\|_{a.s.}$$

for $f_1, \ldots, f_m$ as above. Also, if $x_1^*, \ldots, x_m^*$ are in the unit ball of X^*, then

$$\sum_{n=1}^{m} \left| \int_{O_n} f_n \, d(x_n^* G) \right| = \sum_{n=1}^{m} |x_n^* T(f_n)| \le \sum_{n=1}^{m} \|T(f_n)\| \le \|T\|_{a.s.}$$

Since the $\mathcal{F}_n$'s are "nearly" arbitrary, the regularity of each $x_n^* G$ gives

$$\sum_{n=1}^{m} |x_n^* G|(O_n) \leq \|T\|_{a.s.}$$

(this holds $\forall\, x_1^*, \ldots, x_m^*$ with $\|x_i^*\| \leq 1$).

Now let $E_1, \ldots, E_m$ be disjoint Borel sets. Choose $x_1^*, \ldots, x_m^*$ in X^* s.t. $\|x_i^*\| \leq 1$ and s.t.

$$\sum_{n=1}^{m} |x_n^* G(E_n)| \geq \sum_{n=1}^{m} \|G(E_n)\| - \varepsilon/2$$

By regularity of each $x_n^* G$, we can find compact $F_1, \ldots, F_m$ s.t. $F_i \subseteq E_i$ and

$$\sum_{n=1}^{m} |x_n^* G(F_n)| \geq \sum_{n=1}^{m} |x_n^* G(E_n)| - \varepsilon/2$$

i.e.

$$\sum_{n=1}^{m} |x_n^* G(F_n)| \geq \sum_{n=1}^{m} \|G(E_n)\| - \varepsilon$$

Now $\{F_1, \ldots, F_m\}$ is a disjoint family of compact sets, so there exists disjoint open sets $O_1, \ldots, O_m$ s.t. $F_i \subset O_i$. Accordingly

$$\|T\|_{a.s.} \geq \sum_{n=1}^{m} |x_n^* G|(O_n) \geq \sum_{n=1}^{m} |x_n^* G|(F_n)$$

$$\geq \sum_{n=1}^{m} |x_n^* G(F_n)| \geq \sum_{n=1}^{m} \|G(E_n)\| - \varepsilon$$

This proves $|G|(K) < \infty$ and $|G|(K) \leq \|T\|_{a.s.}$ ☒

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For μ regular and G open

$$|\mu|(G) = \sup_{\substack{\|f\| \le 1 \\ \operatorname{supp} f \subset G}} \left| \int f \, d\mu \right|$$

Proof. Let $\varepsilon > 0$. Write $G = G^+ \cup G^-$. Choose

$$F^+ \subset G^+ \quad |\mu|(G^+ \setminus F^+) < \varepsilon/4$$
$$F^- \subset G^- \quad |\mu|(G^- \setminus F^-) < \varepsilon/4$$

Choose f supported on G, $\|f\| \le 1$ s.t. $f(F^+) = +1$, $f(F^-) = -1$

$$\int_G f \, d\mu = \int_{G^+ \setminus F^+} f \, d\mu + \int_{F^+} f \, d\mu + \int_{F^-} f \, d\mu + \int_{G^- \setminus F^-} f \, d\mu$$

$$= \int_{G^+ \setminus F^+} f \, d\mu + |\mu|(F^+) + |\mu|(F^-) + \int_{G^- \setminus F^-} f \, d\mu$$

$$\ge (-\varepsilon/4) + (|\mu|(G^+) - \varepsilon/4) + (|\mu|(G^-) - \varepsilon/4) + (-\varepsilon/4)$$

$$= |\mu|(G) - \varepsilon$$

Hence $$|\mu|(G) = \sup_{\substack{\|f\| \le 1 \\ \operatorname{supp} f \subset G}} \int f \, d\mu = \sup_{\substack{\|f\| \le 1 \\ \operatorname{supp} f \subset G}} \left| \int f \, d\mu \right|$$

Corollary: An absolutely summing operator of $C(K)$ is weakly compact

Proof. Let $T : C(K) \to X$ be a.s. Then $G << |G|$ and so G is strongly additive, so T is w.c.

$\boxtimes$

Example: Let $\mu \in C(K)^*$. Then the natural embedding $J : C(K) \to L_1(\mu)$ is a.s.

Proof. $J(f) = f \quad \forall f \in C(K)$. Let G = representing measure for J. Then $G(E) = \chi_E \quad \forall E \in \Sigma$. Then

$$|G|(K) = \sup_{\pi} \sum_{E \in \pi} \|G(E)\| = \sup_{\pi} \sum_{E \in \pi} \mu(E) = \mu(K)$$

Hence G is of bounded variation $\Rightarrow$ J a.s. and $\|J\|_{a.s.} = \mu(K)$

Corollary: Let $T : C(K) \to X$ be a bounded linear operator. Then T is a.s. $\Leftrightarrow$ $\exists\, \mu \in C(K)^*$ s.t. T admits the factorization

$$\begin{tikzcd} C(K) \arrow[rr, "T"] \arrow[dr, "J"'] & & X \\ & L_1(\mu) \arrow[ur, "S"'] & \end{tikzcd}$$

Proof. If T admits this factorization, then the fact that J is a.s. and the ideal property of a.s. operators guarantees that $T = SJ$ is a.s.

Conversely, suppose $T : C(K) \to X$ is a.s. with representing measure G. Let $\mu = |G|$. Observe that integration w.r.t. G is a bdd linear operator on $L_1(\mu)$, since

$$\left\| \int_K \sum \alpha_i \chi_{E_i} \, dG \right\| = \left\| \sum \alpha_i G(E_i) \right\| \le \sum |\alpha_i| \, \|G(E_i)\|$$

$$\le \sum |\alpha_i| \, |G|(E_i) = \left\| \sum \alpha_i \chi_{E_i} \right\|_{L_1(\mu)}$$

Call this operator $S : L_1(\mu) \to X$

(Note: IF this case holds, then can select μ s.t. $\mu(K) = \|T\|_{a.s.}$ and $\|S\| \le 1$)

$\square$

<u>**DEFINITION**</u>: Let X, Y be arbitrary Banach spaces. A bounded linear operator $T : X \to Y$ is called Pietsch integral if T admits the following factorization

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "R"'] & Y \\ C(K) \arrow[r, "J"'] & L_1(\mu) \arrow[u, "S"'] \end{tikzcd} \quad \text{for some } \mu \in C(K)^*$$

This should be contrasted with the Grothendieck integral operators which admit the factorization

$$\begin{tikzcd} X \arrow[r] \arrow[d, "R"'] & Y \arrow[r, hook] & Y^{**} \\ C(K) \arrow[r, "J"] & L_1(\mu) \arrow[ur, "S"'] & \end{tikzcd}$$

COROLLARY: Pietsch integral $\Longrightarrow$ a.s. $\Longrightarrow$ w.c.

DEFINITION: A bounded linear operator $T: X \to Y$ is called nuclear if $\exists$ seq. (x_n^*) in X^* and a seq. (y_n) in Y s.t.

(1) $\sum_{n=1}^{\infty} \|x_n^*\| \|y_n\| < \infty$ $\quad (*)$

(2) $Tx = \sum_{n=1}^{\infty} x_n^*(x) y_n \quad \forall x \in X$

In this case $\|T\|_{nc}$ = inf of all sums $(*)$ that obey (2).

Example: Let $\mu \in C(K)^*$. Let $f = \sum_{i=1}^{\infty} x_i \chi_{E_i}$ be a countably disjoint valued function in $L_1(\mu, X)$. Define $T: C(K) \to X$

$$T(g) = \int_\Omega g f \, d\mu$$

i.e.

$$T(g) = \sum_{i=1}^{\infty} x_i \int_{E_i} g \, d\mu = \sum_{i=1}^{\infty} x_i \mu_i(g)$$

$[\mu_i(E) = \mu(E \cap E_i)]$. Hence T satisfies (2). Also

$$\sum_{i=1}^{\infty} \|x_i\| \, \|\mu_i\| = \sum_{i=1}^{\infty} \|x_i\| \mu(E_i) = \|f\|_{L_1(\mu)} < \infty$$

Therefore T is nuclear and

$$\|T\|_{nc} \leq \|f\|_1$$

5/8 VECTOR MEASURES

$T : C(K) \to X$ a.s. $\Leftrightarrow$ G is of bdd variation

THEOREM: $T : C(K) \to X$ is nuclear iff $|G|$ is of bounded variation and G has RN derivative w.r.t. $|G|$. In this case

$$\|T\|_{a.s.} = \|T\|_{n.c.} = |G| = \|g\|_{L_1(|G|, X)}$$

where g is RN derivative.

COROLLARY: X RNP $\Rightarrow$ a.s. $C(K) \to X$ = nucl. op $C(K) \to X$

COROLLARY: Every a.s. $T : C([0,1]) \to X$ nuclear $\Rightarrow$ X has RNP

Proof. If X lacks RNP, then there exist regular G : Borel sets $\to X$ of bounded variation without a derivative. Let

$$T(\cdot) = \int \cdot \, dG$$

on $C[0,1]$. Since $|G|[0,1] < \infty$, we see T is a.s. but not nuclear.

PROPOSITION (Bourgain) Every compact a.s. $T : C([0,1]) \to X$ is nuclear $\Rightarrow$ X has RNP

Proof. Bourgain shows that if all bounded vector measures G with rel. compact range in X have derivatives, then X has RNP.

<u>Proof of theorem</u>: Let (Ω, Σ, μ) be a finite measure space and let $f \in L_1(\mu, X)$, and $\varepsilon > 0$. Then there exist (x_n) in X and (E_n) in Σ such that

1) $f = \sum_{n=1}^{\infty} x_n \chi_{E_n}$ where the converge is a.e. absolute

2) $\int_\Omega \|f\| \, d\mu \leq \sum_{n=1}^{\infty} \|x_n\| \mu(E_n) \leq \int_\Omega \|f\| \, d\mu + \varepsilon$

Take a seq. (g_n) of countably valued functions converging to f uniformly such that WLOG

$$\|f - g_1\|_X \leq \frac{\varepsilon}{2\mu(\Omega)}$$

$$\|g_n - g_{n-1}\| < \frac{\varepsilon}{2^n \mu(\Omega)} \quad n \geq 2$$

Put $g_0 = 0$. Write

$$g_n - g_{n-1} = \sum_{m=1}^{\infty} y_{m,n} \chi_{E_{m,n}}$$

Evidently

$$f = \sum_{n=1}^{\infty} (g_n - g_{n-1}) \quad \text{a.e.}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} y_{m,n} \chi_{E_{n,n}}$$

To test absolute convergence write

$$\sum_n \sum_m \|y_{m,n}\| \chi_{E_{m,n}} \leq \|g_1\|_X + \sum_{n=2}^{\infty} \frac{\varepsilon}{2^n \mu(\Omega)}$$

Integrate

$$\sum_n \sum_m \|y_{m,n}\| \mu(E_{m,n}) \leq \int \|g_1\| d\mu + \varepsilon/2$$
$$\leq \int \|f\| d\mu + \varepsilon.$$

Now write

$$\sum_n \sum_m y_{m,n} \chi_{E_{m,n}} = \sum_{n=1}^{\infty} x_n \chi_{E_n}$$

Now suppose $T: C(K) \to X$ is a.s. Suppose G has an RN derivative g w.r.t. $|G|$. Chose (x_n) in X and (E_n) in Σ s.t.

$$g = \sum x_n \chi_{E_n}$$

where convergence is absolute a.e. and s.t.

$$\int_K \|g\| \, d|G| \leq \sum_{n=1}^{\infty} \|x_n\| \, |G|(E_n) \leq \int_K \|g\| \, d|G| + \varepsilon$$

Observe $f \in C(K) \Rightarrow$

$$T(f) = \int_K f \, dG = \int_K f g \, d|G|$$

$$= \int_K f \sum_n' x_n \chi_{E_n} \, d|G|$$

$$= \sum_{n=1}^{\infty} \left(\int_{E_n} f \, d|G| \right) x_n$$

Recall $S : Y \to Z$ nuclear $\Rightarrow$ $S(y) = \sum_{n=1}^{\infty} y_n^*(y) z_n$, $\sum \|y_n^*\| \|z_n\| < \infty$

$\|S\|_{nc} = \inf$

Let

$$\int_{E_n} (\cdot) \, d|G| = l_n \in C(K)^*$$

We have seen

$$T(f) = \sum_{n=1}^{\infty} l_n(f) x_n$$

Also

$$\sum_{n=1}^{\infty} \|l_n\| \|x_n\| = \sum_{n=1}^{\infty} |G|(E_n) \|x_n\| \leq \int_K \|g\| \, d|G| + \varepsilon < \infty$$

This proves T is nuclear and

$$\|T\|_{nc} \leq \|g\|_{L_1(|G|, X)}$$

Conversely, suppose T is nuclear. Have to show $|G|(K) < \infty$

and $dG/d|G|$ exist. Let $\varepsilon > 0$ and choose (μ_n) in $C(K)^*$ and (x_n) in X s.t.

$$T(f) = \sum_{n=1}^{\infty} \left(\int_K f \, d\mu_n \right) x_n$$

and

$$\sum_{n=1}^{\infty} |\mu_n|(K) \, \|x_n\| \leq \|T\|_{nc} + \varepsilon$$

Put

$$G(E) = \sum_{n=1}^{\infty} x_n \mu_n(E)$$

Then G represents T. G is a c.a. regular X-value measure

$$|G|(K) \leq \sum_{n=1}^{\infty} \|x_n\| \, |\mu_n|(E) \leq \|T\|_{nc} + \varepsilon$$

Therefore $\|T\|_{a.s.} = |G|(K) \leq \|T\|_{nc} + \varepsilon$. All that is left is to produce $dG/d|G|$.

Write

$$G(E) = \sum_{n=1}^{\infty} \bar{\mu}_n(E) x_n + \sum_{n=1}^{\infty} \bar{\bar{\mu}}_n(E) x_n$$

where $\mu_n = \bar{\mu}_n + \bar{\bar{\mu}}_n$ is the Lebesgue decomposition of μ_n w.r.t. $|G|$ s.t. $\bar{\mu}_n \ll |G|$ and $\bar{\bar{\mu}}_n \perp |G|$

$$G(E) = \sum_{n=1}^{\infty} \bar{\mu}_n(E) x_n + \sum_{n=1}^{\infty} \cancelto{0}{\bar{\bar{\mu}}_n}(E) x_n$$

So WLOG $\mu_n << G$. Then

$$G(E) = \sum_{n=1}^{\infty} \mu_n(E) x_n = \sum_{n=1}^{\infty} \int_E f_n \, d|G| \, x_n$$

$f_n \in L_1(|G|)$

$$= \int_E \sum_{n=1}^{\infty} x_n f_n \, d|G|$$

provided $\sum x_n f_n$ is $L_1(|G|, X)$ convergent. To see that $\sum f_n x_n$ is $L_1(|G|, X)$, notice

$$\int_K \left\| \sum_{L=k}^{m} x_n f_n \right\| d|G| = \sum_{n=k}^{m} \int_K |f_n| \|x_n\| \, d|G|$$

$$= \sum_{n=k}^{m} \|x_n\| \, |\mu_n|(K) \longrightarrow 0$$

Hence $\dfrac{dG}{d|G|} = \sum_{n=1}^{\infty} x_n f_n$.

$\blacksquare$

THEOREM (Grothendieck) Let X be a subspace of $L_\infty[0,1]$ that is closed in some $L_p[0,1]$ for $1 \le p < \infty$. Then X is finite dimensional.

Proof. Consider the identity $I : L_\infty \longrightarrow L_p$. I is weakly compact. $I : X_\infty \longrightarrow X_p$ is an isomorphism and so X is reflexive. Let x_n be any bdd seq. in X. Then x_n has a weakly convergent subsequence. Since L_∞ (as a $C(K)$ space) has the Dunford-Pettis property, we see that $I(x_n)$ has a norm convergent subsequence. Since I is an isomorphism, this proves any bdd seq in X has a norm convergent subseq. $\Rightarrow$ X finite dimensional.

$\blacksquare$