WEAK RADON-NIKODYM SETS IN DUAL BANACH SPACES

BY

LAWRENCE HOLLISTER RIDDLE

B.S., Carnegie-Mellon University, 1976
M.S., University of Illinois, 1981

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J. J. Uhl, Jr.
Director of Thesis Research

H. Halberstam
Head of Department

Committee on Final Examination†

Mahlon M. Day, Chairman

J. J. Uhl, Jr.

D. L. Burkholder

N. T. Peck

Robert G. Bartle

† Required for doctor's degree but not for master's.

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CHAPTER 1

Introduction and terminology

The interplay between geometry, topology, measure theory and operator theory has long been evident in the study of the Radon-Nikodým property. Recently, results of substantial interest in the structure of Banach spaces have been obtained by localizing these ideas to individual subsets. The study of the Radon-Nikodým property for subsets of Banach spaces can be thought of as the study of subsets whose structural properties mimic those of the unit ball of a separable dual space.

In this thesis we initiate the study of geometric, topological, measure-theoretic and operator-theoretic characterizations of convex weak*-compact subsets of dual Banach spaces whose structural properties mimic those of the unit ball of the dual of a space that contains no copy of the sequence space ℓ_1 . The search for characterizations of Banach spaces containing no copy of ℓ_1 has been very active since 1974 when Rosenthal [42] proved his striking theorem (for real spaces and later, by Dor, for complex spaces).

Rosenthal's Theorem: *A Banach space X contains no copy of ℓ_1 if and only if every bounded sequence in X has a weakly Cauchy subsequence.*

In Chapter 2 we discuss those global properties of the dual of a Banach space containing no copy of ℓ_1 that we later localize to subsets of dual spaces. At the same time we contrast these properties with the similar, yet subtly different, properties of the duals of Asplund spaces.

Weakly precompact sets occupy our attention in Chapter 3. Because of Rosenthal's Theorem, weak precompactness may be considered as a localization of the property of containing no copy of ℓ_1. After some initial comments on weakly precompact sets, we use weak precompactness in $L_\infty(\mu)$ to identify dual-valued Pettis integrable functions and Pettis representable operators. As one consequence of this work we are able to show that bounded universally scalarly measurable functions on a compact Hausdorff space taking values in the dual of a separable space are Pettis integrable with respect to all regular Borel measures on the domain space.

Chapter 4 contains the main part of the thesis. Here we localize the ideas in Chapter 2 to obtain various types of characterizations for weak*-compact absolutely convex sets whose structural properties mimic those of the unit ball of the dual of a space not containing ℓ_1. The cornerstone for our results is a factorization theorem that will allow us to make use of weakly precompact sets and Rosenthal's Theorem to obtain characterizations in terms of Dunford-Pettis operators, universal measurability and points of continuity of linear functionals in the second dual space, Baire-1 functions, Rademacher trees, extreme points and dentability.

The last chapter studies an interesting property for families of real-valued functions that was originally formulated by Jean Bourgain. This Bourgain property essentially allows us to replace the pointwise convergence of a net of functions by an almost everywhere sequential convergence. Our main result is that a bounded dual-valued function $f : \Omega \to X^*$ is Pettis integrable if the family $\{\langle f,x\rangle : ||x|| \leq 1\}$ has the Bourgain property. We also discuss how the Bourgain property identifies Pettis representable operators.

It is now time to fix some terminology. For the most part our notation and terminology will confirm with that found in the bibles of Banach space and vector measure theory - Dunford-Schwartz [11] and Diestel-Uhl [10]. Throughout this thesis X and Y are real Banach spaces with duals X^* and Y^*, respectively. By the unit ball, B_X, of X we will always mean the closed unit ball. Similarly, an operator will always be a bounded linear operator.

The triple (Ω,Σ,μ) will always be a finite measure space. A function f from Ω into X is <u>strongly measurable</u> if there is a sequence (s_n) of simple functions from Ω into X such that

$$\lim_n ||f(t) - s_n(t)|| = 0$$

for almost all t in Ω. If f is strongly measurable and there is a sequence (s_n) of simple functions such that

$$\lim_n \int_\Omega ||f - s_n|| \, d\mu = 0 ,$$

then f is said to be Bochner integrable and we define

$$\int_E f\, d\mu = \lim_n \int_E s_n\, d\mu ,$$

where the integral of a simple function is defined in the usual way.

A function $f : \Omega \to X$ is scalarly measurable if the scalar function $x^*f(\cdot)$ is measurable for each x^* in the dual space X^*. In addition, the function f is Pettis integrable [33] if for each set E in Σ there is an element x_E of X that satisfies

$$x^*(x_E) = \int_E x^*f\, d\mu$$

for every x^* in X^*. In this case we write $x_E = \text{Pettis} - \int_E f\, d\mu$. The Pettis norm of a Pettis integrable function f is defined to be $\sup \{\int |x^*f| d\mu : x^* \in X^* , ||x^*|| \leq 1\}$.

A dual-valued function $f : \Omega \to X^*$ is weak*-scalarly measurable if the scalar function $\langle f(\cdot),x\rangle$ is measurable for each x in X. If, in addition, the function $\langle f(\cdot),x\rangle$ is integrable for each x in X, in particular if f is bounded, then an easy closed graph argument (see Diestel and Uhl [10, page 53]) produces for each E in Σ an element x_E^* in X^* such that

$$x_E^*(x) = \int_E \langle f,x\rangle\, d\mu$$

for each x in X. The element x_E^* is called the Gel'fand (or weak*-)

integral of f over the set E.

If Ω is a compact Hausdorff space, then a real-valued function ϕ defined on Ω is universally measurable if ϕ is μ-measurable for every Radon probability measure μ on Ω; that is, if and only if there exists, for each regular probability measure μ on the Borel σ-algebra and for each $\alpha > 0$, a compact subset E of Ω with $\mu(\Omega \setminus E) < \alpha$ such that the restriction of ϕ to E is continuous. If f is a function from Ω into a Banach space X, then f is called universally scalarly measurable if the real-valued function $x^*f(\cdot)$ is universally measurable for each x^* in X^*. In addition, the function f is called universally Pettis integrable if it is μ-Pettis integrable for each Radon probability measure μ.

An operator $T : L_1(\mu) \to X$ is said to be Bochner [respectively, Pettis] representable if there is a Bochner [respectively, Pettis] integrable function $f : \Omega \to X$ such that

$$T(g) = \int_\Omega gf \, d\mu$$

[respectively, $T(g) = \text{Pettis} - \int_\Omega gf \, d\mu$] for each function g in $L_1(\mu)$. The Banach space X is said to have the Radon-Nikodým property [respectively, weak Radon-Nikodým property] if for every finite measure space (Ω,Σ,μ), every operator $T : L_1(\mu) \to X$ is Bochner [respectively, Pettis] representable. Finally, an operator $T : X \to Y$ is said to be a Dunford-Pettis operator if it maps weakly compact sets into norm compact sets.

A function f is said to be Baire-1 if f is the pointwise limit of a sequence of continuous functions. In 1899 Baire [2] published the following remarkable result about such functions:

The Baire Characterization Theorem: Let K be a non-empty compact metric space and f a real-valued function defined on K. Then f is a Baire-1 function if and only if for every non-empty closed subset M of K, the restriction of f to M has a point of continuity relative to the topological space M.

Finally, a Banach space contains no copy of ℓ_1 if it has no subspace homeomorphic to the usual sequence space ℓ_1.

Parts of this thesis have appeared in [35], [36], [38] and [39].

CHAPTER 2

Global properties of the dual of a Banach space containing no copy of ℓ_1

In this chapter we discuss those properties of the dual of a Banach space containing no copy of ℓ_1 that we shall subsequently localize to weak*-compact convex subsets of dual spaces. Until 1974 it was thought by many that a separable Banach space that contains no copy of ℓ_1 must have a separable dual. In that year, James [23] put matters straight with one of his celebrated spaces, the James Tree space, thus demonstrating that the class of separable spaces with separable duals and the class of separable spaces containing no copy of ℓ_1 are not identical classes.

This fact notwithstanding, there are many ways in which these two classes are similar, yet subtly different. During the course of this chapter we shall occasionally illustrate these differences. For this reason, most of the chapter is expository and proofs will, in general, not be included since the proofs for the local results will be given later.

Let us agree that a Banach space is an Asplund space if each of its separable subspaces has a separable dual. The connection between Asplund spaces and differentiation of measures has been studied by a host of authors whose cumulative efforts revealed that a Banach space is an Asplund space if and only if its dual has the Radon-Nikodým property for the Bochner integral. The history of this theorem is chronicled in Diestel and Uhl [10].

Shortly thereafter, Musial [27] showed that a separable Banach space contains no copy of ℓ_1 if and only if its dual has the Radon-Nikodým property for the Pettis integral. His proof used a deep result of Odell and Rosenthal [31] on the weak*-sequential denseness in X^{**} of a separable space X containing no copy of ℓ_1. Subsequently, Janicka [24] employed Haydon's non-separable analogue of the Odell-Rosenthal theorem to extend Musial's characterization to arbitrary Banach spaces. Bourgain [3] independently obtained the same result by combining a property of families of real-valued functions with Rosenthal's Theorem in order to replace pointwise net convergence of functions with almost everywhere sequential convergence (see Chapter 5). We see, therefore, that the difference between the class of Asplund spaces and the class of spaces not containing a copy of ℓ_1 is the difference between the Bochner and Pettis integrals.

Radon-Nikodým theorems for vector measures also yield representation results for operators on L_1. Another operator-theoretic characterization of spaces not containing ℓ_1 may be given in terms of Dunford-Pettis operators.

Pelczynski's Theorem: Any one of the following statements about a Banach space X implies all the others.

(a) The space X contains no copy of ℓ_1.

(b) Every bounded linear operator from $L_1[0,1]$ into X^* is a Dunford-Pettis operator.

(c) The dual X^* contains no copy of $L_1[0,1]$.

Proof. To prove that (a) implies (b), let $S : L_1[0,1] \to X^*$ be a bounded linear operator. Because $L_1[0,1]$ is separable, the closure of the range of S is a separable subspace Z of X^*. An appeal to a standard trick of Dunford and Schwartz (see [11, VI.8.8] or [10, III.3.6]) produces a separable subspace Y of X such that Z is isometric to a subspace of Y^*. Note that Y contains no copy of ℓ_1. Thus we can assume that S is a bounded linear operator from $L_1[0,1]$ into Y^*. Next, the separability of Y and an easy compactness argument originally due to Dunford and Pettis (see [11, VI.8.6] or the first part of the proof of [10, III.3.1]) produces a bounded function $g: [0,1] \to Y^*$ such that

$$S(f)y = \int_{[0,1]} f(t)\langle g(t),y\rangle \, dt$$

for all f in $L_1[0,1]$ and for all y in Y. To show that S maps weakly compact sets into norm compact sets, it is enough to show that S acts as a (norm) compact operator from $L_\infty[0,1]$ into Y^*. To understand why, note that a weakly compact set K in $L_1[0,1]$ is uniformly integrable and, in particular, that

$$\lim_n \int |f|\chi_{[|f| > n]} \, dt = 0$$

uniformly in f belonging to K. Since

$$S(K) \subset \{S(f\chi_{[|f| \le n]}): f \in K\} + \{S(f\chi_{[|f| > n]}): f \in K\} ,$$

and if S maps $L_\infty[0,1]$-bounded sets into compact sets, then it easily follows that $S(K)$ is totally bounded.

Towards showing that S is a compact operator on $L_\infty[0,1]$, define an operator $R : Y \longrightarrow L_1[0,1]$ by

$$R(y)(t) = \langle g(t), y \rangle$$

for y in Y and t in $[0,1]$. Let (y_n) be a bounded sequence in Y. Since Y contains no copy of ℓ_1, Rosenthal's Theorem guarantees that (y_n) has a weakly Cauchy subsequence, which we also call (y_n). A glance at the definition of R shows that $R(y_n)$ is a pointwise Cauchy sequence and hence converges pointwise to a measurable function. Moreover, the boundedness of g and the boundedness of (y_n) guarantee that $(R(y_n))$ is $L_\infty[0,1]$-bounded. This, combined with the bounded convergence theorem, proves that $\lim_n R(y_n)$ exists in the $L_1[0,1]$-norm. Hence $R : Y \longrightarrow L_1[0,1]$ is a compact operator and so is $R^* : L_\infty[0,1] \longrightarrow Y^*$. But now a routine computation shows that $R^*(f) = S(f)$ for each f in $L_\infty[0,1]$. Hence S acts as a compact operator on $L_\infty[0,1]$. This proves that (a) implies (b).

The proof that (b) implies (c) is trivial. If X^* contains a copy of $L_1[0,1]$, then take any isomorphism $S : L_1[0,1] \longrightarrow X^*$ and notice that S can not take weakly compact sets into norm compact sets.

That (c) implies (a) is well-known and easy. If X contains a copy, Y, of ℓ_1, then there is an isomorphism between ℓ_∞ and

$X^*/Y^\perp$. Since ℓ_∞ contains copies of $L_1[0,1]$, there is an isomorphism $S : L_1[0,1] \longrightarrow X^*/Y^\perp$. By an old theorem of Grothendieck [20, Proposition 1], there is an operator $R : L_1[0,1] \to X^*$ such that $S = \sigma R$ where σ is the quotient map of X^* onto $X^*/Y^\perp$. It follows directly that R is an isomorphism and this completes the proof.

There are two ways to use martingales to highlight the difference between Asplund spaces and spaces not containing a copy of ℓ_1 . The relationship between the Radon-Nikodým property and martingale mean convergence emerged from a number of independent papers in the 1960's including Chatterji [6], Metivier [26], Rønnow [41] and Uhl [49, 50]. What is known is that a Banach space X is an Asplund space (i.e., X^* has the Radon-Nikodým property) if and only if every $L_\infty([0,1],X^*)$-bounded martingale (f_n,B_n) is Cauchy in the Bochner norm; i.e.,

$$\lim_{n,m} \int_{[0,1]} ||f_n(t) - f_m(t)|| \, dt = 0 .$$

The related result for spaces not containing ℓ_1 has also been around for a long time. It is a direct consequence of material from Pelczynski's Theorem, Musial [27] and Uhl [50], and says that a Banach space X contains no copy of ℓ_1 if and only if every $L_\infty([0,1],X^*)$-bounded martingale (f_n,B_n) is Cauchy in the Pettis norm; i.e.,

$$\lim_{n,m} \sup_{\|x\| \le 1} \int_{[0,1]} |f_n(t)x - f_m(t)x| \, dt = 0 \, .$$

In this instance, therefore, we see that the difference between the class of Asplund spaces and the class of spaces not containing a copy of ℓ_1 is the difference between the Bochner and Pettis norm.

A similar situation exists for the pointwise convergence of martingales. A Banach space X is an Asplund space if and only if for every $L_1([0,1],X^*)$-bounded uniformly integrable martingale (f_n,B_n) there is a function f on $[0,1]$ with values in X^* and a fixed null set E such that $\lim_n x^{**}f_n(t) = x^{**}f(t)$ for all x^{**} in X^{**} and for all t not in E. On the other hand, a Banach space contains no copy of ℓ_1 if and only if for every $L_1([0,1],X^*)$-bounded uniformly integrable martingale (f_n,B_n) there is a function f on $[0,1]$ with values in X^{**} such that $\lim_n x^{**}f_n(t) = x^{**}f(t)$ almost everywhere for each x^{**} in X^{**}, where here the exceptional set can vary with x^{**} in X^{**}. Thus from the point of view of pointwise convergence of martingales, the difference between the class of Asplund spaces and the class of spaces not containing a copy of ℓ_1 is the difference between fixed null sets and mobile null sets.

Stegall (see [10, VII.2.6]) has given a characterization of Asplund spaces in terms of trees in the dual. In view of the close relation between trees and martingales, it is perhaps not surprising that Banach spaces not containing a copy of ℓ_1 can be characterized in terms of trees in the dual.

A sequence (x_n) in a Banach space X is called a tree if

$$x_n = (x_{2n} + x_{2n+1})/2$$

for all $n = 1,2,\ldots$. The martingale (f_n) associated with a tree (x_n) is defined by

$$f_1 = x_1\chi_{[0,1]} ,$$

$$f_2 = x_2\chi_{[0,\frac{1}{2}]} + x_3\chi_{[\frac{1}{2},1]} ,$$

$$f_3 = x_4\chi_{[0,\frac{1}{4}]} + x_5\chi_{[\frac{1}{4},\frac{1}{2}]} + x_6\chi_{[\frac{1}{2},3/4]} + x_7\chi_{[3/4,1]} ,$$

etc.

A tree (x_n) is a δ-tree if there exists a $\delta > 0$ such that $||x_n - x_{2n}|| \geq \delta$ and $||x_n - x_{2n+1}|| \geq \delta$ for all $n = 1,2,\ldots$. One of the prime facts about δ-trees is that the martingale associated with a δ-tree can not be Cauchy in the Bochner norm [10, page 125]. This is the underlying reason for why a Banach space is an Asplund space if and only if its dual contains no bounded δ-tree. For a history of this result, whose finished version is due to Stegall, consult [10, Chapters 3 and 7].

Now to obtain a characterization of Banach spaces X containing no copy of ℓ_1 , it is natural to ask what conditions on a tree (x_n^*) in X^* guarantee that the martingale (f_n) associated with (x_n^*) is not Cauchy in the Pettis norm. The idea is to find a tree that attacks at the brittle roots of the weak Radon-Nikodým property as it spirals around the unit ball of X^* (for related phenomena see

J. Fletcher et al [14]). One way to do this is to force

$$\sup_{||x||\leq 1} \int_{[0,1]} |f_{n+1}(t)x - f_n(t)x| \, dt \geq \delta$$

for all $n = 1,2,\ldots$ and for some fixed $\delta > 0$. A brief computation shows that this amounts to

$$||x_1^*|| \geq \delta ,$$

$$||x_2^* - x_3^*|| \geq 2\delta ,$$

$$||x_4^* - x_5^* + x_6^* - x_7^*|| \geq 4\delta ,$$

etc.

A tree (x_n^*) satisfying the above inequalities is called a δ-<u>Rademacher tree</u>. Two of the best known trees are both Rademacher trees. Setting $x_1 = \chi_{[0,1]}$, $x_2 = 2\chi_{[0,\frac{1}{2}]}$, $x_3 = 2\chi_{[\frac{1}{2},1]}$, $x_4 = 4\chi_{[0,\frac{1}{4}]}$, $x_5 = 4\chi_{[\frac{1}{4},\frac{1}{2}]}$, etc., produces a 1-Rademacher tree in $L_1[0,1]$. Also, letting $x_1 = (1,0,0,0,\ldots)$, $x_2 = (1,1,0,0,\ldots)$, $x_3 = (1,-1,0,0,\ldots)$, $x_4 = (1,1,1,0,\ldots)$, $x_5 = (1,1,-1,0,\ldots)$, $x_6 = (1,-1,1,0,\ldots)$, $x_7 = (1,-1,-1,0,\ldots)$, etc., yields a tree in the sequence space c_0 that is easily seen to be a 1-Rademacher tree. In view of the discussion above and Pelczynski's Theorem, it becomes clear that a Banach space contains no copy of ℓ_1 if and only if its dual contains no bounded δ-Rademacher tree.

In 1936 J. A. Clarkson defined the geometric notion of uniform convexity for the purpose of proving a Radon-Nikodým theorem for the Bochner integral. Then after a thirty year lapse, Rieffel [40] rekindled the belief in the Radon-Nikodým property as an internal geometric property of Banach spaces by introducing the concept of dentability. We say that a bounded subset A of a dual space X^* is <u>weak*-dentable</u> if for every $\alpha > 0$ there exists a weak*-open slice

$$S = \{x^* \in A : x^*(x) > \sup_{y^* \in A} y^*(x) - \beta\}$$

for some x in X and $\beta > 0$ such that the norm diameter of S is less than α. In 1975 Namioka and Phelps [30] proved that a Banach space X is an Asplund space if and only if every bounded (weak*-compact convex) subset of X^* is weak*-dentable. Then in 1981 Elias and Paulette Saab [45] defined a bounded subset A of X^* to be <u>weak*-scalarly dentable</u> if for every $\alpha > 0$ and every x^{**} in X^{**} there exists a weak*-open slice S of the form above such that the diameter of $x^{**}S$ is less than α; i.e.,

$$\sup \{|x^{**}(x^*) - x^{**}(y^*)| : x^*, y^* \in S\} < \alpha .$$

They proceeded to show that X contains no copy of ℓ_1 if and only if every bounded (weak*-compact convex) subset of X^* is weak*-scalarly dentable.

By now it is well known that duals of Asplund spaces have incredibly strong extremal properties. Indeed, a series of papers culminating in the work of Huff and Morris [22] showed that a Banach space X is an Asplund space if and only if every norm-closed bounded convex subset of X^* is the norm-closed convex hull of its extreme points. On the other hand, Haydon [21] showed that a Banach space X contains no copy of ℓ_1 if and only if every weak*-compact convex subset of X^* is the norm-closed convex hull of its extreme points. In this context, therefore, the difference between the class of Asplund spaces and the class of spaces not containing a copy of ℓ_1 is the difference in extremal properties of bounded convex subsets of the dual that are norm-closed but not weak*-compact.

The final criterion that we shall discuss concerns continuity and measurability conditions on functionals in the second dual of a Banach space. Following Saab and Saab [45], we shall say that a dual space X^* has the <u>scalar point of continuity property</u> if for each weak*-compact subset M of the unit ball of X^* and for each x^{**} in X^{**}, the restriction of x^{**} to the set M equipped with the weak*-topology has a point of continuity. It is easy to see that any dual space X^* has the scalar point of continuity property if the restriction of every x^{**} in X^{**} to the unit ball of X^* with the weak*-topology is a Baire-1 function. The Baire Characterization Theorem [2] ensures that the converse holds when X is separable. Odell and Rosenthal [31] used this fact to show that a separable Banach space X does not contain a copy of ℓ_1 if and only if the restriction of every x^{**} in X^{**} to the unit ball of X^* is a Baire-1 function. Then in 1981 Saab and Saab [45] showed that any Banach space (separable or not) contains no copy of ℓ_1 if and only if its dual has the scalar point of continuity property.

In attempting to find a non-separable analogue of the Odell-Rosenthal result, Haydon [21] considered their ideas in terms of measurability rather than of the first Baire class and showed that a Banach space X contains no copy of ℓ_1 if and only if every element of X** is universally measurable as a function on the unit ball of X* equipped with the weak*-topology. Another way of expressing this is to say that X contains no copy of ℓ_1 if and only if the natural identity map from the unit ball of X* with the weak*-topology into X* is universally scalarly measurable. It should be remarked that work by Schwartz [46] and by Saab [44] shows that a Banach space X is an Asplund space if and only if this map is universally measurable.

CHAPTER 3

Weakly precompact sets

One useful method for investigating a property of Banach spaces is to localize the property by defining and studying it for an individual subset. Thus one encounters, for example, Dunford-Pettis sets, Radon-Nikodým sets, and recently Stegall's GSP sets, which are in some sense a localization of the Asplund property. In view of the subtle differences between Asplund spaces and spaces not containing ℓ_1, it is not surprising that GSP sets are closely related to weakly precompact sets. In the next chapter we shall prove an analogue of a factorization theorem of Stegall which shows, roughly, that weakly precompact sets are to sets with the weak Radon-Nikodým property as GSP sets are to sets with the Radon-Nikodým property. First, however, we take a closer look at weakly precompact sets and their role in recognizing Pettis integrable functions and Pettis representable operators.

Definition: (a) A subset B of a Banach space X is called weakly precompact if every bounded sequence in B has a weakly Cauchy subsequence.

(b) We say that a bounded sequence (x_n) in X is a copy of the ℓ-basis if there exists $\delta > 0$ such that

$$||\sum a_k x_k|| \geq \delta \sum |a_k|$$

for all finitely non-zero sequences (a_k) of reals.

In his fundamental paper [42], Rosenthal proved that a bounded subset of X is weakly precompact if and only if it contains no copy of the ℓ_1-basis. In particular, the space X contains no copy of ℓ_1 if and only if the unit ball of X is weakly precompact.

Theorem 1 : *A bounded subset B of a Banach space X is weakly precompact if and only if for each finite measure space (Ω,Σ,μ), each bounded linear operator $S : X \longrightarrow L_\infty(\mu)$ takes sequences in B into sequences with almost everywhere convergent subsequences.*

Proof. First suppose that B is weakly precompact. Let $S : X \longrightarrow L_\infty(\mu)$ be a bounded operator and let (x_n) be a sequence in B. Without loss of generality, we may assume that X is separable. By [11, VI.8.6] there exists a bounded weak*-measurable function $g : \Omega \longrightarrow X^*$ such that the restriction of S^* to $L_1(\mu)$ satisfies

$$S^*f(x) = \int_\Omega f\langle g,x\rangle \, d\mu \quad \text{for all} \quad f \in L_1(\mu)$$

for each x in X. It follows that $Sx(\cdot) = \langle g(\cdot),x\rangle$ almost everywhere for each x in X, and therefore that there exists a null set E satisfying

$$Sx_n(\omega) = \langle g(\omega),x_n\rangle \quad \text{for all} \quad n$$

for each ω in $\Omega\setminus E$. Now invoke the weak precompactness of B to find a weakly Cauchy subsequence (x_{n_k}) of (x_n) and observe that $\langle g(\cdot), x_{n_k} \rangle$ converges pointwise. Hence (Sx_{n_k}) converges almost everywhere.

Conversely, suppose B is not weakly precompact and use Rosenthal's Theorem to find a sequence (x_n) in B that is a copy of the ℓ_1-basis. Let Z be the closed linear span of the sequence (x_n). Letting r_n denote the n^{th} Rademacher function, define S from Z to $L_\infty[0,1]$ by

$$S(\sum a_n x_n) = \sum a_n r_n \ .$$

Because (x_n) is a copy of the ℓ_1-basis, the operator S is bounded, and since $L_\infty[0,1]$ is an injective space, the operator S has a bounded extension to all of X. However, $Sx_n = r_n$ for each n and the sequence of Rademacher functions has no almost everywhere convergent subsequence. This completes the proof.

Stegall [48] defined a GSP set as follows: a subset B of a Banach space X has the GSP if, for each finite measure space (Ω, Σ, μ) and each bounded linear operator $S : X \longrightarrow L_\infty(\mu)$, the set $S(B)$ is equimeasurable; that is, for each $\alpha > 0$ there is a measurable set A such that $\mu(\Omega \setminus A) < \alpha$ and the set $\{S(x)\chi_A : x \in B\}$ is a relatively norm compact subset of $L_\infty(\mu)$. It follows that if B is a GSP set, then for each operator $S : X \longrightarrow L_\infty(\mu)$ one can find a set E of small measure such that for each sequence (x_n) in B, the sequence $(Sx_n \chi_{\Omega\setminus E})$ has an $L_\infty(\mu)$-convergent subsequence. On the other hand, if B is weakly precompact, then Egorov's Theorem ensures that for each sequence (x_n) in B, there is a set E of small measure, with E depending on the particular sequence chosen, such that $(Sx_n \chi_{\Omega\setminus E})$ has an $L_\infty(\mu)$-convergent subsequence. The distinction between weakly precompact sets and GSP sets is thus the distinction between mobile exceptional sets and stationary exceptional sets.

Observations of Odell and Stegall [43] yield another characterization of weakly precompact sets, namely that a subset B of a Banach space X is weakly precompact if and only if for every Banach space Y and for every Dunford-Pettis operator $S : X \longrightarrow Y$, the set $S(B)$ is a relatively compact subset of Y. It follows immediately from this characterization that the closed convex hull of a weakly precompact set is also weakly precompact.

Geitz and Uhl [18] used the topological property of weak compactness in $L_\infty(\mu)$ and $B(\Sigma)$ to study Pettis integrable functions and scalarly measurable functions that are weakly equivalent to strongly measurable functions. Several other authors, including Phillips [34], Edgar [12, 13], Fremlin-Talagrand [16], Geitz [17] and Sentilles-Wheeler [47], have also studied the problem of recognizing Pettis integrable functions. By using weak precompactness in $L_\infty(\mu)$ and a deep theorem of Bourgain, Fremlin and Talagrand, we shall show that bounded universally scalarly measurable functions on a compact Hausdorff space taking values in the dual of a separable Banach space are universally Pettis integrable. The next lemma points in this direction.

Lemma 2 : Let (Ω,Σ,μ) be a finite measure space and let X be a separable Banach space. If $f : \Omega \longrightarrow X^*$ is a bounded weak*-scalarly measurable function with the property that for each $\alpha > 0$ there is a set E in Σ with $\mu(\Omega\setminus E) < \alpha$ and such that the set $\{\langle f,x\rangle\chi_E : ||x|| \leq 1\}$ is weakly precompact in $L_\infty(\mu)$, then f is μ-Pettis integrable.

Proof. Let $\alpha > 0$ and take such a set E. Define $T : X \longrightarrow L_\infty(\mu)$ by $Tx = \langle f,x\rangle\chi_E$ and note that the hypothesis guarantees that $T(B_X)$ is weakly precompact. According to the factorization construction [9] of Davis, Figiel, Johnson and Pelczynski, the operator T factors through a Banach space containing no copy of ℓ_1. Because of the duality between spaces not containing ℓ_1 and dual spaces with the weak Radon-Nikodým property, we see that the adjoint operator $T^* : (L_\infty(\mu))^* \longrightarrow X^*$ factors through a space with the weak Radon-Nikodým property. In particular, the operator $T^* : L_1(\mu) \longrightarrow X^*$ factors through a space with the weak Radon-Nikodým property. Accordingly, there is a bounded Pettis integrable function $g : \Omega \longrightarrow X^*$ such that

$$T^*\phi = \text{Pettis} - \int_\Omega \phi g \, d\mu$$

for every ϕ in $L_1(\mu)$. A moment's reflection shows that

$$\langle f,x\rangle\chi_E = Tx = \langle g,x\rangle \quad \text{a.e.}$$

for each x in X. Because X is separable, it follows that $f\chi_E = g$ almost everywhere and hence that $f\chi_E$ is Pettis integrable. A standard exhaustion argument shows that f is itself Pettis integrable (see, for example, [37, page 23]).

Theorem 3 : Let X be a separable Banach space and let K be a compact Hausdorff space. If $f : K \longrightarrow X^*$ is a bounded universally scalarly measurable function, then f is universally Pettis integrable.

Proof. Let $\alpha > 0$ and fix a Radon probability measure μ on K. Let (x_n) be a dense subset of the unit ball of X. For each integer n there exists a set E_n in Σ such that $\mu(\Omega \setminus E_n) < \alpha/2^n$ and such that the restriction of $\langle f,x_n\rangle$ to E_n is continuous. Let $E = \bigcap_{n=1}^{\infty} E_n$; then $\mu(\Omega \setminus E) < \alpha$ and the restriction $\langle f,x_n\rangle|E$ of $\langle f,x_n\rangle$ to E is continuous for each integer n. Because the sequence (x_n) is dense in the unit ball of X, the triangle inequality ensures that $\langle f,x\rangle|E$ is continuous for all x in the unit ball of X.

Let $A = \{\langle f,x\rangle|E : ||x|| \leq 1\} \subset C(E)$ and let $M_r(E)$ be the set of all real-valued universally measurable functions on E equipped with the topology of pointwise convergence. Let $f_\eta = \langle f,x_\eta\rangle|E$ be a net in A and choose a subnet (x_β) of (x_η) that converges weak* to some x^{**} in X^{**}. If $\phi = x^{**}f|E$, then ϕ belongs to $M_r(E)$ and $f_\beta(t)$ converges to $\phi(t)$ for every t in E. Therefore A is a relatively compact subset of $M_r(E)$. By a direct application of a theorem of Bourgain, Fremlin and Talagrand [5, Theorem 2F], every sequence in A has a pointwise convergent subsequence; i.e., this set is weakly precompact in $C(E)$. Because the ℓ_1-basis is not weakly Cauchy, the set A can not contain a copy of the ℓ_1-basis in $C(E)$, and since the inclusion map of $C(E)$ into $L_\infty(K,\mu)$ is a contraction, we see that the set $\{\langle f,x\rangle\chi_E : ||x|| \leq 1\}$ contains no copy of the ℓ_1-basis in the $L_\infty(K,\mu)$ norm. Hence this set is weakly precompact in $L_\infty(K,\mu)$. Appeal to Lemma 2 to complete the proof.

For a separable X , the easiest way to show that a bounded function $f : K \longrightarrow X^*$ is universally Pettis integrable is to show that $x^{**}f$ is Borel measurable for each x^{**} in X^{**} . We shall do this for the following well-known examples.

Example 4 : For this example we consider Hagler's "Murphy's Pub Function" [10] on the unit interval. Let (A_n) denote the sequence of dyadic intervals of $[0,1]$ obtained by setting $A_1 = [0,1]$, $A_2 = [0,\frac{1}{2}]$, $A_3 = [\frac{1}{2},1]$, $A_4 = [0,\frac{1}{4}]$, $A_5 = [\frac{1}{4},\frac{1}{2}]$, etc. Define the function $f : [0,1] \longrightarrow \ell_\infty$ by $f(t) = (\chi_{A_n}(t))$ for each t in $[0,1]$.

Let ϕ be in the unit ball of ℓ_∞^* and let ϕ_1 be the countably additive part of ϕ and let ϕ_2 be the purely finitely additive part of ϕ . (Here ϕ_1 is the countably additive measure on the subsets of the integers given by $\phi_1(E) = \sum_{n\in E} \phi(\{n\})$ for each subset E of the integers.) Now

$$\phi(f(t)) = \phi_1(f(t)) + \phi_2(f(t))$$

$$= \sum_{n=1}^{\infty} [\chi_{A_n}(t)\phi(\{n\})] + \phi_2(f(t)) \ .$$

In Diestel and Uhl [10, page 43] it is shown that $\phi_2(f(\cdot))$ is countably non-zero. Accordingly, the function $\phi(f(\cdot))$ is Borel measurable and therefore f is universally Pettis integrable by Theorem 3.

Example 5 : Define the function $f : [0,1] \longrightarrow L_\infty[0,1]$ by $f(t) = \chi_{[0,t]}$ for each t in $[0,1]$. Take μ in the unit ball of $L_\infty[0,1]^*$ and write $\mu = \mu^+ - \mu^-$ where μ^+ and μ^- are the non-negative measures in $L_\infty[0,1]^*$ obtained from the Jordon Decomposition Theorem. Fix a t in $[0,1]$ and observe that

$$\begin{aligned} \langle \mu, f(t) \rangle &= \int_0^1 f(t)\, d\mu \\ &= \int_0^1 f(t)\, d\mu^+ - \int_0^1 f(t)\, d\mu^- \\ &= \int_0^1 \chi_{[0,t]}\, d\mu^+ - \int_0^1 \chi_{[0,t]}\, d\mu^- \\ &= \mu^+([0,t]) - \mu^-([0,t]) \ . \end{aligned}$$

Because this latter expression is the difference of two monotonic functions of t , we see that $\langle \mu, f(\cdot) \rangle$ is Borel measurable. Therefore f is universally Pettis integrable.

It would be gratifying to be able to remove the separability hypothesis in the statement of Theorem 3. However, it is impossible to dispense with it entirely because if $X = \ell_1[0,1]$, then Phillips has given an example [34] of a bounded function $f : [0,1] \longrightarrow \ell_\infty[0,1]$ such that $x^{**}f$ is constant almost everywhere for each x^{**} in X^{**} , and hence Borel measurable, but such that f is not integrable with respect to Lebesgue measure.

This leaves one obvious question unresolved: What happens when X is weakly compactly generated? Unfortunately we do not have a satisfactory answer. The best we can offer is the following fact.

Corollary 6 : Let X be a WCG Banach space and let K be a compact Hausdorff space. If $f : K \longrightarrow X^*$ is a bounded universally scalarly measurable function whose range is weak*-separable, then f is universally Pettis integrable.

Proof. This is an easy consequence of Theorem 3 and a theorem of Amir-Lindenstrauss (see Day [8, page 74]). Select a weak*-separable subspace M of X^* that contains the range of f . Use the theorem of Amir-Lindenstauss to write $X = X_1 + X_2$ and $X^* = X_1^* + X_2^*$ where X_1 is separable and $M \subset X_1^*$. By Theorem 3, the function f is universally Pettis integrable into X_1^* and hence into X^* . This completes the proof.

The best way to view Corollary 6 is in light of Pettis's Measurability Theorem [10, II.1.2] which says that if f is universally scalarly measurable and has a norm separable range, then f is universally Bochner integrable. Corollary 6 merely relaxes the norm separability condition and guarantees (the weaker) universal Pettis integrability.

Before passing to the next section we pause to ask several questions. First, is the converse of Lemma 2 true? Fremlin and Talagrand have given an example [16] of a Pettis integrable function into ℓ_∞ for which (as was pointed out to us by G. A. Edgar) the converse of Lemma 2 is not true. Their example, however, does not a priori rule out the converse of Lemma 2 because their underlying measure is not a Radon measure. It should be observed, though, that the global converse of Lemma 2 fails, as the following example demonstrates.

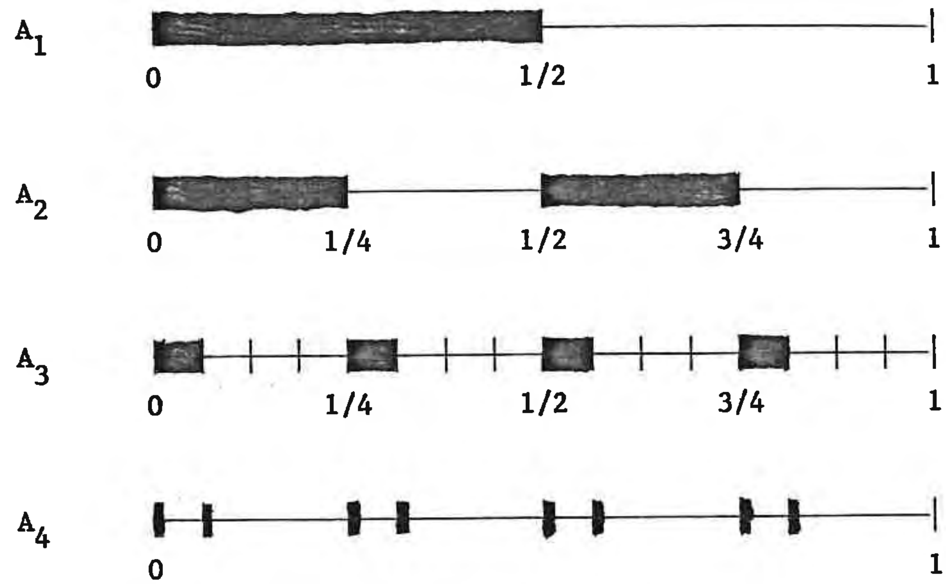
Example 7 : We shall define a sequence of subsets of $[0,1]$ as follows: let $A_1 = [0,\frac{1}{2})$; if A_m is the union of disjoint intervals

$$A_m = \bigcup_{i=1}^{2^{m-1}} [a_i,b_i) ,$$

then define the set A_{m+1} by

$$A_{m+1} = \bigcup_{i=1}^{2^{m-1}} \left([a_i,a_i + \frac{1}{2^{2^m}}) \cup [b_i,b_i + \frac{1}{2^{2^m}})\right) .$$

(see diagram next page)



Since $\lambda(A_{m+1}) = 2^m/2^{2^m}$ for each integer m (where λ is Lebesgue measure), we see that $\sum \lambda(A_{m+1}) < \infty$. Therefore, the Borel-Cantelli Lemma implies that the sequence (χ_{A_m}) converges almost everywhere to 0 .

We now will show that the sequence (χ_{A_m}) is equivalent in $L_\infty[0,1]$ to the ℓ_1-basis by using an argument due to Rosenthal [42]. Let $B_m = [0,1]\setminus A_m$ and observe that

$$\lambda\Big(\bigcap_{m\in P} A_m \cap \bigcap_{m\in N} B_m\Big) > 0$$

for all finite disjoint subsets P and N of integers. Let (a_k) be a sequence of reals with only finitely many non-zero terms. We must show that there exists a constant β , independent of the sequence (a_k) , such that

$$\beta \sum |a_k| \le \|\sum a_k \chi_{A_k}\|_\infty .$$

Without loss of generality, we may assume that $\sum |a_k| = 1$. Let E be an arbitrary null subset of $[0,1]$. Let $P = \{k : a_k \ge 0\}$ and let $N = \{k : a_k < 0\}$. By our observation above, we may choose

$$t_1 \in \Big(\bigcap_{k\in P} A_k \cap \bigcap_{k\in N} B_k\Big)\setminus E ,$$

$$t_2 \in \Big(\bigcap_{k\in N} A_k \cap \bigcap_{k\in P} B_k\Big)\setminus E .$$

Then

$$\sum a_k \chi_{A_k}(t_1) = \sum_{k\in P} a_k \chi_{A_k}(t_1) + \sum_{k\in N} a_k \chi_{A_k}(t_1)$$

$$= \sum_{k\in P} a_k = \sum_{k\in P} |a_k| .$$

Similarly, $-\sum a_k \chi_{A_k}(t_2) = -\sum_{k\in N} a_k = \sum_{k\in N} |a_k|$. Consequently,

$$\sum a_k \chi_{A_k}(t_1) - \sum a_k \chi_{A_k}(t_2) = \sum |a_k| = 1 ,$$

so that either

$$|\sum a_k \chi_{A_k}(t_1)| \ge \tfrac{1}{2} \quad \text{or} \quad |\sum a_k \chi_{A_k}(t_2)| \ge \tfrac{1}{2} .$$

For either case we obtain

$$\sup_{t \notin E} \left|\sum a_k \chi_{A_k}(t)\right| \geq \tfrac{1}{2} .$$

Since E was an arbitrary null set, this last inequality shows that

$$\left\|\sum a_k \chi_{A_k}\right\|_\infty \geq \tfrac{1}{2} .$$

Finally, define a function $f : [0,1] \longrightarrow \ell_\infty$ by $f(t) = (\chi_{A_m}(t))$ for each t in $[0,1]$. The first part of the example shows that f is scalarly measurable and essentially separably valued. By Pettis's Measurability Theorem, therefore, the function f is strongly measurable. Moreover,

$$\int_0^1 \|f(t)\|_\infty \, dt \leq 1 < \infty$$

and hence f is even Bochner integrable. The second part, however, shows that the set $\{\langle f,x\rangle : x \in \ell_1 , \|x\| \leq 1\}$ is not weakly precompact in $L_\infty[0,1]$ since it contains the sequence (χ_{A_n}).

As a preliminary to our second question, define a Banach space X to have the universal Pettis integral property (UPIP) if every bounded scalarly measurable function with values in X is universally Pettis integrable. According to Theorem 3, if X is the dual of a separable space, the X has the UPIP . What are the spaces with the UPIP ? Do set-theoretic axioms play the important role in the study of the UPIP that they play in the study of the stronger

Pettis integral property (PIP) as found in Edgar [12, 13] and Fremlin-Talagrand [16]?

We now turn our attention to applications of weakly pre-compact subsets of $L_\infty(\mu)$ in recognizing operators on $L_1(\mu)$ that are representable by Pettis integrable functions. First suppose that $T : L_1(\mu) \longrightarrow X^*$ is a Dunford-Pettis operator. Then the restricted operator $T : L_2(\mu) \longrightarrow X^*$ is compact [4] and therefore $T^* : X \longrightarrow L_2(\mu)$ is also compact. It follows easily from this observation that the operator $T : L_1(\mu) \longrightarrow X^*$ is a Dunford-Pettis operator if and only if $T^* : X \longrightarrow L_\infty(\mu)$ takes bounded sequences into sequences with almost everywhere convergent subsequences. On the other hand, it is well known [10, III.2.21] that $T : L_1(\mu) \longrightarrow X^*$ is Bochner representable if and only if the set $T^*(B_X)$ is almost relatively weakly compact in $L_\infty(\mu)$; i.e., for each $\alpha > 0$ there exists a set E in Σ with $\mu(\Omega \setminus E) < \alpha$ such that $T^*(B_X)\chi_E$ is relatively weakly compact. The topological property in $L_\infty(\mu)$ characterizing Pettis representable operators from $L_1(\mu)$ into X^* should therefore fall somewhere between weak convergence and almost everywhere convergence. In light of Theorem 1 the next result should not be surprising.

<u>Theorem 8</u> : <u>An operator</u> $T : L_1(\mu) \longrightarrow X^*$ <u>is Pettis representable if the set</u> $T^*(B_X)$ <u>is almost weakly precompact in</u> $L_\infty(\mu)$; <u>i.e., for each</u> $\alpha > 0$ <u>there is a set</u> E <u>in</u> Σ <u>with</u> $\mu(\Omega \setminus E) < \alpha$ <u>such that</u> $T^*(B_X)\chi_E$ <u>is weakly precompact</u>.

<u>Proof</u>. Let $\alpha > 0$ and take such a set E. Define another operator $S : L_1(\mu) \longrightarrow X^*$ by $S(f) = T(f\chi_E)$ for each f in $L_1(\mu)$. A moment's reflection reveals that $S^*x = (T^*x)\chi_E$ for each x in X and consequently $S^*(B_X)$ is weakly precompact in $L_\infty(\mu)$. As in the proof of Lemma 2, the operator $S^{**} : L_1(\mu)^{**} \longrightarrow X^*$ factors through a space with the weak Radon-Nikodým property. In particular, the operator $S : L_1(\mu) \longrightarrow X^*$ is Pettis representable. A standard exhaustion argument then shows that T is Pettis representable and completes the proof.

We conjecture that the converse of Theorem 8 is also true, at least for perfect measure spaces. To understand why this is reasonable, examine the following diagram for an operator $T : L_1(\mu) \to X^*$.

T Bochner representable	$\Longleftrightarrow$	$T^*(B_X)$ almost rel. weakly compact
$\Downarrow$		$\Downarrow$
T Pettis representable	$\Longleftarrow$	$T^*(B_X)$ almost weakly precompact
$\Downarrow$		$\Downarrow$
T Dunford-Pettis	$\Longleftrightarrow$	$T^*(B_X)$ "a.e. relatively compact"

If (Ω,Σ,μ) is a perfect measure space [15], then Stegall [16] has shown that the vector measure $F(E) = T(\chi_E)$ associated with a Pettis representable operator T on $L_1(\mu)$ has a relatively compact range. It follows that T is a Dunford-Pettis operator [see Observation 4.2 , page 41]. This establishes the bottom left implication in the above diagram for perfect measure spaces.

To see why the bottom right implication holds, observe that Theorem 1 ensures thay every sequence in a weakly precompact subset of $L_\infty(\mu)$ (for an arbitrary measure μ) has an almost everywhere convergent subsequence. Now find an increasing sequence of measurable sets E_n such that $\mu(\cup E_n) = \mu(\Omega)$ and such that $T^*(B_X)\chi_{E_n}$ is weakly precompact in $L_\infty(\mu)$. Then a routine diagonalization argument produces an almost everywhere convergent subsequence of a sequence (f_n) in $T^*(B_X)$.

The global converse of Theorem 8 does fail, however. Let $f : [0,1] \longrightarrow \ell_\infty$ be the Bochner integrable function constructed in Example 7 and define a Bochner representable operator $T : L_1[0,1] \to \ell_\infty$ by $T(\phi) = \int_0^1 \phi(t)f(t)\,dt$ for each ϕ in $L_1[0,1]$. Then $T^* : \ell_1 \longrightarrow L_\infty[0,1]$ fixes a copy of ℓ_1 since $T^*e_m = \chi_{A_m}$ for each integer m .

For operators into ℓ_∞ , a special situation arises because of the existence of coordinate functions. If $T : L_1(\mu) \longrightarrow \ell_\infty$ is a bounded linear operator, it is easy to see that there is a uniformly bounded sequence (g_n) of functions in $L_\infty(\mu)$ such that

$$T\phi = \left(\int_\Omega \phi g_n \, d\mu\right)$$

for each ϕ in $L_1(\mu)$. Now suppose that this sequence is almost weakly precompact in $L_\infty(\mu)$; that is, for each $\alpha > 0$ there is a set

E in Σ with $\mu(\Omega \setminus E) < \alpha$ and such that the set $\{g_n \chi_E : n \in N\}$ is weakly precompact in $L_\infty(\mu)$. Define a function $g : \Omega \longrightarrow \ell_\infty$ by $g(t) = (g_n(t))$ for each t in Ω . Let (e_n) denote the usual unit-vector basis in ℓ_1 . Since $\langle g, e_n \rangle = g_n$ for each integer n , it is clear that g is weak*-scalarly measurable. Now let $\alpha > 0$ and take a set E satisfying the above hypothesis. Observe that the set $A = \{\langle g, \pm e_n \rangle \chi_E : n \in N\}$ is weakly precompact in $L_\infty(\mu)$. Therefore the closed convex hull of A is also weakly precompact in $L_\infty(\mu)$ and consequently the set $\{\langle g, x \rangle \chi_E : x \in \overline{co}(\pm e_n)\}$ is also weakly precompact in $L_\infty(\mu)$. The Krein-Mil'man Theorem now ensures that the set $\{\langle g, x \rangle \chi_E : ||x|| \leq 1\}$ is weakly precompact in $L_\infty(\mu)$. Finally, appeal to Lemma 2 to conclude that the function g is Pettis integrable and then glance at the definition of g to see that T is Pettis representable with kernel g . The next theorem summarizes this discussion.

Theorem 9 : Let $T : L_1(\mu) \longrightarrow \ell_\infty$ be a bounded linear operator with coordinate functionals (g_n) . If the sequence (g_n) is almost weakly precompact in $L_\infty(\mu)$, then T is Pettis representable.

We conjecture that the converse of Theorem 9 also holds, at least for perfect measure spaces (such as [0,1], for example). To understand why, consider the corresponding situation for Bochner representable operators and Dunford-Pettis operators into ℓ_∞ .

Theorem 10 : *Let* $T : L_1(\mu) \longrightarrow \ell_\infty$ *have the representation*

$$T\phi = \left(\int_\Omega \phi g_n \, d\mu\right)$$

where (g_n) *is a uniformly bounded sequence in* $L_\infty(\mu)$.

(a) *The operator* T *is Bochner representable if and only if the sequence* (g_n) *is almost relatively weakly compact in* $L_\infty(\mu)$.

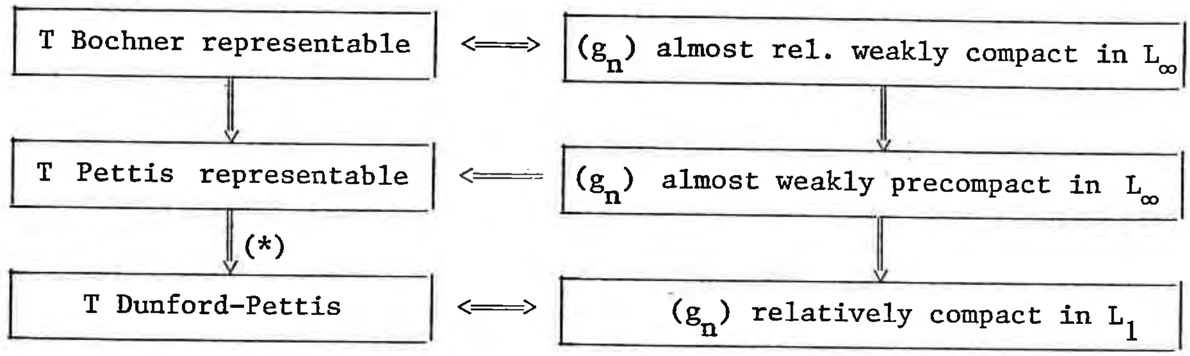
(b) *The operator* T *is a Dunford-Pettis operator if and only if the sequence* (g_n) *is relatively (norm) compact in* $L_1(\mu)$.

Proof. A quick calculation reveals that $T^*x = \sum x_n g_n$ for each x in the ball of ℓ_1 . Therefore $T^*(B_{\ell_1})$ is contained in the closed convex hull of $\{\pm g_n : n \in N\}$. If the sequence (g_n) is almost relatively weakly compact in $L_\infty(\mu)$, then the set $T^*(B_{\ell_1})$ is also almost relatively weakly compact in $L_\infty(\mu)$ and consequently T is Bochner representable; but (g_n) is clearly contained in $T^*(B_{\ell_1})$, so the converse implication also holds. This establishes part (a).

For part (b), recall that if T is a Dunford-Pettis operator, then T^* maps bounded sequences in ℓ_1 into sequences with almost everywhere convergent subsequences. Therefore each subsequence of (g_n) has an $L_1(\mu)$-convergent subsequence; that is, the sequence (g_n) is relatively compact in $L_1(\mu)$. Conversely, if (g_n) is relatively compact in $L_1(\mu)$, then as above, the set $T^*(B_{\ell_1})$ is also relatively compact in $L_1(\mu)$. Therefore T^* maps bounded sequences into sequences with $L_1(\mu)$-convergent subsequences, which in turn have almost everywhere convergent subsequences. Hence T is a Dunford-Pettis operator and this completes the proof.

The following chart summarizes the previous discussion.

$$T : L_1(\mu) \longrightarrow \ell_\infty \qquad T\phi = \left(\int_\Omega \phi g_n \, d\mu\right)$$



(*) for perfect measure spaces

Let $f : (\Omega,\Sigma,\mu) \longrightarrow Y$ be a Pettis integrable function into a Banach space Y and let Γ be a sub-σ-algebra of Σ. A Pettis integrable function $g : (\Omega,\Gamma,\mu) \longrightarrow Y$ is said to be a <u>Pettis conditional expectation</u> of f with respect to the σ-algebra Γ if g is scalarly Γ-measurable and if

$$\text{Pettis} - \int_E g \, d\mu = \text{Pettis} - \int_E f \, d\mu$$

for each set E in Γ. Our last theorem in this chapter provides a sufficient condition for a bounded dual-valued Pettis integrable function to have Pettis conditional expectation. It naturally makes use of weakly precompact sets.

Theorem 11 : Let $f : (\Omega,\Sigma,\mu) \longrightarrow X^*$ be a bounded Pettis integrable function. If the set $\{\langle f,x\rangle : \|x\| \leq 1\}$ is weakly precompact in $L_\infty(\mu)$, then f has Pettis conditional expectation with respect to all sub-σ-algebras of Σ.

Proof. Let Γ be a sub-σ-algebra of Σ and define an operator $T : X \longrightarrow L_\infty(\Gamma,\mu)$ by

$$Tx = E(\langle f,x\rangle|\Gamma)$$

for each x in X. Since the set $\{\langle f,x\rangle : \|x\| \leq 1\}$ contains no copy of the ℓ_1-basis in $L_\infty(\Sigma,\mu)$ and the conditional expectation operator E is a contraction from $L_\infty(\Sigma,\mu)$ into $L_\infty(\Gamma,\mu)$, we may conclude that $T(B_X)$ contains no copy of the ℓ_1-basis in $L_\infty(\Gamma,\mu)$. Consequently $T(B_X)$ is weakly precompact in $L_\infty(\Gamma,\mu)$ and an appeal to Lemma 8 produces a Pettis integrable kernel $g : (\Omega,\Gamma,\mu) \longrightarrow X^*$ for the operator $T^* : L_1(\Gamma,\mu) \longrightarrow X^*$. Then

$$\langle g,x\rangle = Tx = E(\langle f,x\rangle|\Gamma) \quad \text{a.e.}$$

for every x in X. Therefore

$$\int_B \langle g,x\rangle \, d\mu = \int_B E(\langle f,x\rangle | \Gamma) \, d\mu = \int_B \langle f,x\rangle \, d\mu$$

for every set B in Γ and hence

$$\text{Pettis} - \int_B g \, d\mu = \text{Pettis} - \int_B f \, d\mu$$

for every set B in Γ. This shows that g is a Pettis conditional expectation of f for the σ-algebra Γ.

Unfortunately, the converse of Theorem 11 is false, since Example 7 exhibits a Bochner integrable function $f : [0,1] \longrightarrow \ell_\infty$, which thus has conditional expectations, but for which the sequence $(\langle f,e_m\rangle) = (\chi_{A_m})$ is equivalent to the ℓ_1-basis.

CHAPTER 4

Weak Radon-Nikodým sets
in dual Banach spaces

Shortly after the appearance of Rosenthal's signal theorem on spaces containing ℓ_1, a number of additional characterizations of such spaces appeared. They were discussed in Chapter 2 and are collected below for reference.

Theorem 1 : Each of the following statements characterizes Banach spaces X that contain no copy of ℓ_1.

(a) (Haydon) Every x^{**} in X^{**} is universally (weak*-) measurable and satisfies the barycentric formula on the unit ball of X^* equipped with the weak*-topology.

(b) (Pelczynski) Every bounded linear operator $T : L_1 \longrightarrow X^*$ is a Dunford-Pettis operator.

(c) (Musial-Janicka) The dual X^* possesses the Radon-Nikodým property for the Pettis integral.

(d) (Saab and Saab) The restriction of each x^{**} in X^{**} to each non-empty weak*-compact subset of X^* has a point of weak*-continuity.

(e) (Riddle and Uhl) The dual X^* does not contain a Rademacher tree.

(f) (Haydon) Every weak*-compact convex subset of X^* is the norm-closed convex hull of its extreme points.

(g) (Saab and Saab) Every non-empty bounded set in X^* is weak*-scalarly dentable.

The goal of this chapter is to localize these theorems by showing that statements (a) through (g) above localize to provide equivalent conditions for absolutely convex weak*-compact subsets of dual Banach spaces. By and large the work is based on an analogue of a factorization theorem of Stegall that says, roughly, that weakly precompact sets are to sets with the weak Radon-Nikodým property as GSP sets are to sets with the Radon-Nikodým property.

§1 The Factorization Theorem

Throughout this section let $T : X \longrightarrow Y$ be a fixed bounded linear operator between the Banach spaces X and Y.

In [48] Stegall proved that $T(B_X)$ is a GSP set in Y if and only if $T^*(B_{Y^*})$ is a Radon-Nikodým set in X^*; that is, for every finite measure space (Ω,Σ,μ) and every bounded linear operator $S : L_1(\mu) \longrightarrow X^*$ for which $S(\chi_E/\mu(E))$ belongs to $T^*(B_{Y^*})$ for every set E in Σ of positive measure, the operator S is represented by a Bochner integrable function. In this section we shall show that an analogous statement can be made for weakly precompact sets in Y and weak Radon-Nikodým sets in X^*.

Let us first fix some terminology. A subset K of X is called a <u>weak Radon-Nikodým set</u> [respectively, a <u>set of complete continuity</u>] if for every finite measure space (Ω,Σ,μ) and every bounded linear operator $S : L_1(\mu) \longrightarrow X$ for which $S(\chi_E/\mu(E))$ belongs to K for each non-null set in Σ, the operator S is represented by a

Pettis integrable function with values in K [respectively, is a Dunford-Pettis operator]. A weak Radon-Nikodým set is also said to have the weak Radon-Nikodým property.

There is an alternative description of sets of complete continuity in terms of vector measures. It is based on the following well-known observation [10, pages 92-95 and page 161].

Observation 2 : Any one of the following statements about an operator $S : L_1(\mu) \longrightarrow X$ implies all the others.

(a) The operator S is a Dunford-Pettis operator.

(b) The restriction of S to $L_\infty(\mu)$ defines a compact operator from $L_\infty(\mu)$ to X.

(c) The vector measure F defined for E in Σ by $F(E) = S(\chi_E)$ has relatively norm compact range.

The average range of a vector measure $F : \Sigma \longrightarrow X$ is defined to the set

$$\left\{\frac{F(E)}{\mu(E)} : E \in \Sigma \text{ , } \mu(E) > 0\right\} .$$

The observation ensures that a set K is a set of complete continuity if and only if for each finite measure space (Ω,Σ,μ) and each μ-continuous vector measure $F : \Sigma \longrightarrow X$ of bounded variation with average range contained in K, the measure F has a relatively norm compact range. The global property in this vector measure context has been studied by Musial in [27].

Haydon [21] showed that if a Banach space X contains no copy of ℓ_1, then every x^{**} in X^{**} is universally measurable and satisfies the barycentric formula on the unit ball of X^* eqipped with the weak*-topology. This means that if μ is a Radon measure on the unit ball of X^* equipped with the weak*-topology, then there is an element x_μ^* in X^* such that

$$x^{**}(x_\mu^*) = \int_{B_{X^*}} x^{**}(x^*)\, d\mu(x^*)$$

for all x^{**} in X^{**}. In the terminology of the Introduction, this just means that the identity operator $I : (B_{X^*}, \text{weak*}) \longrightarrow X^*$ is universally Pettis integrable. Using this fact, Janicka [24] showed that if ℓ_1 does not embed in X, then B_{X^*} is a weak Radon-Nikodým set. Replacing B_{X^*} by a weak*-compact convex subset C of X^* in Janicka's proof yields the following lemma.

<u>Lemma 3</u> : <u>Let</u> X <u>be a Banach space and let</u> C <u>be a weak*-compact convex subset of</u> X^* <u>such that every</u> x^{**} <u>in</u> X^{**} <u>is universally measurable and satisfies the barycentric formula on</u> C <u>equipped with the weak*-topology</u>. <u>Then</u> C <u>is a weak Radon-Nikodým set</u>.

The next theorem is an analogue of a theorem of Stegall [48, Theorem 1.12] and forms the basis for most of this chapter.

Theorem 4 : Each of the following statements about an operator $T : X \longrightarrow Y$ implies all the others.

(a) The set $T(B_X)$ is weakly precompact.

(b) The operator T factors through a Banach space that contains no copy of ℓ_1.

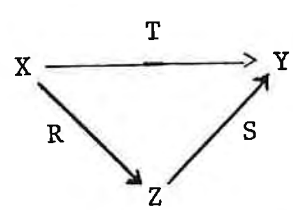
(c) The set $T^*(B_{Y^*})$ is a weak Radon-Nikodým set.

(d) The set $T^*(B_{Y^*})$ is a set of complete continuity.

(e) The adjoint operator T^* factors through a Banach space with the weak Radon-Nikodým property.

Proof. (a) $\Rightarrow$ (b). This follows from the factorization construction of Davis, Figiel, Johnson and Pelczynski. According to [9, Lemma 1(xii)], if $T(B_X)$ is weakly precompact, then the unit ball of the space constructed in [9] is also weakly precompact. In light of Rosenthal's Theorem, this means that T factors through a space containing no copy of ℓ_1.

(b) $\Rightarrow$ (c). By the preceding remarks and Lemma 3, it suffices to show that the identity map $I : (T^*(B_{Y^*}), \text{weak*}) \longrightarrow X^*$ is universally Pettis integrable. Towards this end, suppose T admits the factorization



where the Banach space Z contains no copy of ℓ_1 . Then we have $T^*(B_{Y^*}) = R^*(S^*(B_{Y^*}))$. By Haydon [21], the identity map from the unit ball of Z^* with the weak*-topology into Z^* is universally Pettis integrable. Consider the following factorization

$$\begin{array}{ccc} (S^*(B_{Y^*}),\ \text{weak*}) & \xrightarrow{\ J\ } & (Z^*,\ \|\cdot\|) \\ {\scriptstyle R_1^*}\big\downarrow & & \big\downarrow{\scriptstyle R^*} \\ (T^*(B_{Y^*}),\ \text{weak*}) & \xrightarrow{\ I\ } & (X^*,\ \|\cdot\|) \end{array}$$

where $R_1^*(z^*) = R^*(z^*)$ for each z^* in $S^*(B_{Y^*})$ and I and J are the respective identity maps. Observe that R_1^* is continuous. Also, J is universally Pettis integrable and thus R^*J is universally Pettis integrable.

Let μ be a Radon probability measure on $(T^*(B_{Y^*}),\ \text{weak*})$. Using the fact that $(S^*(B_{Y^*}),\ \text{weak*})$ is compact and R_1^* is onto, one can easily produce a Radon probability measure υ on the space $(S^*(B_{Y^*}),\ \text{weak*})$ such that $R_1^*(\upsilon) = \mu$ [1, page 90]. But $R^*J = IR_1^*$ is υ-scalarly measurable and thus I is $R_1^*(\upsilon) = \mu$-scalarly measurable. Moreover, $R^*J = IR_1^*$ is υ-Pettis integrable. Employ a change-of-variables formula to see that I is $R_1^*(\upsilon) = \mu$-Pettis integrable.

(c) $\Rightarrow$ (d) . Let (Ω,Σ,μ) be a finite measure space and let $S : L_1(\mu) \longrightarrow X^*$ be an operator for which $S(\chi_E/\mu(E))$ belongs to $T^*(B_{Y^*})$ for every E in Σ of positive measure. Since ℓ_1 has the Schur property, we may assume without loss of generality that μ

is non-atomic. To show that S is a Dunford-Pettis operator, it suffices to show that the vector measure $F(E) = S(\chi_E)$ defined on Σ has relatively norm compact range. Towards this end, take a sequence (E_n) of measurable sets. There exists an isomorphism β mapping the σ-algebra generated by the sequence (E_n) onto the σ-algebra B of Borel subsets of $[0,1]$ with Lebesgue measure. Define a vector measure $G : B \longrightarrow X^*$ by $G(E) = F(\beta^{-1}(E))$. Then the weak Radon-Nikodým set $T^*(B_{Y^*})$ contains the average range of G, so the measure G can be represented as an indefinite Pettis integral. Since the unit interval with the Borel subsets is a perfect measure space, Stegall's Theorem [16, Proposition 3J] ensures that G has a relatively norm compact range. Therefore $F(E_n) = G(\beta(E_n))$ has a convergent subsequence in the dual X^*, and so we have shown that F has a relatively norm compact range.

(d) $\Rightarrow$ (a). Suppose $T(B_X)$ is not weakly precompact and use Rosenthal's Theorem to find a sequence (Tx_n) that is a copy of the ℓ_1-basis (e_n). Let Y_o denote the closed linear span of the sequence (Tx_n). Then the definition $S(Tx_n) = e_n$ gives rise to an isomorphism S from Y_o onto ℓ_1. Let (e_n^*) be the usual ℓ_∞-basis and set $y_n^* = S^*(e_n^*)$. Then $y_n^*(Tx_k) = \delta_{k,n}$ (Kronecker delta). If r_k is the k^{th} Rademacher function, then the definition of a function $f_n : [0,1] \longrightarrow Y_o^*$ by

$$f_n(\cdot) = ||S||^{-1} \sum_{k=1}^{n} r_k(\cdot) y_k^*$$

produces a martingale (f_n) with respect to the dyadic partitions of the unit interval $[0,1]$. Now the operator $R_o : L_1[0,1] \longrightarrow Y_o^*$ given by $R_o(g) = \lim_n \int g f_n \, d\lambda$ (λ is Lebesgue measure on $[0,1]$) has a norm-preserving lifting [29, Theorem 2] to an operator $R : L_1[0,1] \to Y^*$; i.e., $i^*R = R_o$, where $i : Y_o \longrightarrow Y$ is the natural injection, and $||R|| = ||R_o||$. Then if π_n denotes the dyadic partition of $[0,1]$ into intervals of length $1/2^n$, the functions

$$g_n(\cdot) = \sum_{A \in \pi_n} \frac{R(\chi_A)}{\lambda(A)} \chi_A(\cdot)$$

form a martingale from $[0,1]$ into Y^* satisfying

$$f_n(\cdot)y = g_n(\cdot)y \quad \text{for all} \quad y \in Y_o$$

$$\sup_n ||f_n||_\infty = \sup_n ||g_n||_\infty .$$

Therefore B_{Y^*} contains $g_n(\Omega)$, and so (T^*g_n) is an X^*-valued martingale with $T^*g_n(\Omega)$ contained in $T^*(B_{Y^*})$. However, computing the Pettis norm of $T^*g_{n+1} - T^*g_n$, we obtain

$$\begin{aligned} &\sup \{ \int |\langle x^{**}, T^*g_{n+1} - T^*g_n \rangle| \, d\lambda : ||x^{**}|| < 1\} \\ &\geq \int |\langle T^*g_{n+1} - T^*g_n, x_{n+1} \rangle| \, d\lambda \\ &= \int |\langle f_{n+1} - f_n, Tx_{n+1} \rangle| \, d\lambda \\ &= ||S||^{-1} \int |r_{n+1}| \, d\lambda = ||S||^{-1} . \end{aligned}$$

Therefore (T^*g_n) is not Cauchy in the Pettis norm. This means that the operator $W : L_1[0,1] \longrightarrow X^*$ given by $W(f) = \lim_n \int f(T^*g_n)\, d\lambda$ is not a Dunford-Pettis operator [4]. Since $W(\chi_E/\lambda(E))$ does belong to $T^*(B_{Y^*})$ for each dyadic interval E, we see that $T^*(B_{Y^*})$ is not a set of complete continuity.

(b) $\Rightarrow$ (e) . The dual of a space without any copy of ℓ_1 has the weak Radon-Nikodým property by Janicka's theorem [24] .

(e) $\Rightarrow$ (a) . Suppose $T(B_X)$ is not weakly precompact. Let (g_n) be the martingale constructed during the proof of the penultimate implication. If T^* factors through a space with the weak Radon-Nikodým property, then (T^*g_n) converges in the Pettis norm [28]. However, we saw that (T^*g_n) is not even Cauchy in the Pettis norm, and consequently condition (e) must fail.

This completes the proof.

Any weak*-compact absolutely convex set K in the dual X^* of a Banach space X can be written in the form $K = T^*(B_{Y^*})$. To see this, just take Y to be the space $C(K)$ of continuous functions on K and let $T : X \longrightarrow Y$ be the evaluation operator defined by $Tx(x^*) = x^*(x)$. Because the closed unit ball of $C(K)^*$ is just the weak*-closure of the convex hull of the unit point masses, it is not difficult to see that $T^*(B_{Y^*}) = K$. This observation will allow us to use Theorem 4 to study arbitrary weak*-compact absolutely convex subsets of dual spaces.

The following corollary is the main result of this section.

In subsequent sections we shall find additional characterizations of weak*-compact subsets that are weak Radon-Nikodým sets.

Corollary 5 : An absolutely convex weak*-compact subset of a dual space is a weak Radon-Nikodým set if and only if it is a set of complete continuity.

Proof. This is an immediate consequence of the preceding observation and Theorem 4.

Corollary 6 : An absolutely convex weak*-compact subset of a dual space is a weak Radon-Nikodým set if and only if it has the weak Radon-Nikodým property for the unit interval.

Proof. Let K be an absolutely convex weak*-compact subset of a dual space X^* and let (Ω,Σ,μ) be an arbitrary finite measure space. Let $S : L_1(\mu) \longrightarrow X^*$ be an operator with $S(\chi_E/\mu(E))$ in K for each set E of positive measure. Let $F : \Sigma \longrightarrow X^*$ be the vector measure defined by $F(E) = S(\chi_E)$ for each E in Σ. To show that K is a weak Radon-Nikodým set, it suffices by Corollary 5 and Observation 2 to show that F has a relatively norm compact range. Towards this end, let (E_n) be a sequence of sets in Σ. Because ℓ_1 has the Schur property, we may assume without loss of generality that μ is non-atomic. Then there exists an isomorphism β mapping the σ-algebra generated by the sequence (E_n) onto the σ-algebra B of Borel subsets of the unit interval. Define a vector measure $G : B \longrightarrow X^*$ by $G(E) = F(\beta^{-1}(E))$. Then the set K contains the average range of G. Because K has the weak Radon-Nikodým property on the unit interval, Stegall's theorem (as in the proof of (c) => (d) in Theorem 4) ensures that G has a relatively norm compact range. Consequently, the sequence $F(E_n) = G(\beta(E_n))$ has a convergent subsequence in X^*.

Corollary 7 : Let K be a weak*-compact absolutely convex weak Radon-Nikodým subset of a dual space X^*. For every bounded linear operator S from a Banach space Z into X, the set $S^*(K)$ is a weak Radon-Nikodým set in Z^*.

Proof. Write $K = T^*(B_{Y^*})$ for an operator $T : X \longrightarrow Y$. By Theorem 4, the operator T factors through a space not containing a copy of ℓ_1. But $S^*(K) = (TS)^*(B_{Y^*})$ and TS clearly factors through the same space containing no copy of ℓ_1 as does T. Another appeal to Theorem 4 reveals that $S^*(K)$ is a weak Radon-Nikodým set.

§2 Measurability, continuity and weak Radon-Nikodým sets

In the previous section we saw that if the identity map on a weak*-compact convex subset of X^* is universally Pettis integrable, then that set has the weak Radon-Nikodým property. Having seen earlier how to recognize when a function if universally Pettis integrable, we now combine these results toward obtaining additional characterizations of weak Radon-Nikodým sets.

Theorem 8 : For a separable Banach space X and a weak*-compact convex subset C of X^*, the following statements are equivalent.

(a) The set C has the weak Radon-Nikodým property.

(b) The identity map $I : (C, \text{weak*}) \longrightarrow X^*$ is universally scalarly measurable.

(c) The identity map $I : (C, \text{weak*}) \longrightarrow X^*$ is universally Pettis integrable.

(c′) Every x^{**} in X^{**} is universally measurable and satisfies the barycentric formula on $(C, \text{weak*})$.

Proof. Because X is separable, Theorem 3.3 guarantees the equivalence of (b) and (c). That (c) implies (a) is just Lemma 3. Thus all we need to show is that (a) implies (c). Let μ be a Radon probability measure on $(C, \text{weak*})$ and let Σ be the σ-algebra of Borel subsets of $(C,\text{weak*})$. For every function f in $L_1(C,\mu)$, define $S(f)$ in X^* by letting $S(f)$ be the Gelfand integral of the function $f(\cdot)I(\cdot) : C \longrightarrow X^*$; that is,

$$S(f)x = \int_C f\langle I,x\rangle \, d\mu$$

for each x in X. Then $S : L_1(C,\mu) \longrightarrow X^*$ defines a bounded linear operator with $S(\chi_E/\mu(E))$ belonging to C for each E in Σ of positive measure. Because C is a weak Radon-Nikodým set, there exists a μ-Pettis integrable function $g : C \longrightarrow X^*$ such that for each f in $L_1(C,\mu)$ we have $S(f) = \text{Pettis} - \int_C gf\, d\mu$. Therefore, for each x in X and each E in Σ we see that

$$\int_E \langle I,x\rangle \, d\mu = S(\chi_E)x = \int_E \langle g,x\rangle \, d\mu \, .$$

Because X is separable, these last equalities imply that $I = g$ μ-almost everywhere and hence that I is μ-Pettis integrable. This completes the proof.

Theorem 8 yields a Choquet type representation with the Pettis integral. Let C be a weak*-compact convex weak Radon-Nikodým subset of the dual X^* of a separable space X , and let x^* be an element of C . Choquet's Theorem [7] ensures the existence of a Borel probability measure μ on C which is supported on the extreme points of C and satisfies

$$x^*(x) = \int_C y^*(x) \, d\mu(y^*) \quad \text{for every } x \in X \, ;$$

that is, x^* is the Gel'fand or weak*-integral of the identity map $I : (C, \text{weak}^*) \longrightarrow X^*$ with respect to the measure μ . By Theorem 8, however, the map I is μ-Pettis integrable and therefore x^* is actually represented as a Pettis integral with respect to μ ; i.e.,

$$x^{**}(x^*) = \int_C x^{**}(\cdot) \, d\mu \quad \text{for every } x^{**} \in X^{**} \, .$$

For an arbitrary Banach space X , the conclusion of Theorem 8 still holds if C is absolutely convex. Before showing this, however, we pause to consider the following concept introduced by Saab and Saab in [45].

Definition : Let A be a weak*-compact subset of X^* . We say that A has the scalar point of continuity property (SPCP) if for each weak*-compact subset M of A and each x^{**} in X^{**} , the restriction of x^{**} to $(M, \text{weak*})$ has a point of continuity.

It was shown by Saab and Saab [45] that a Banach space does not contain a copy of ℓ_1 if and only if every weak*-compact set in X^* has the scalar point of continuity property. The next theorem localizes their result and localizes Haydon's theorem as mentioned in the discussion before Lemma 3.

Theorem 9 : Let X be a Banach space and A a weak*-compact absolutely convex subset of X^* . Then each of the following statements about A implies all the others.

(a) The set A has the weak Radon-Nikodým property.

(b) The set A has the scalar point of continuity property.

(c) The identity map $I : (A, \text{weak*}) \longrightarrow X^*$ is universally scalarly measurable.

(d) The identity map $I : (A, \text{weak*}) \longrightarrow X^*$ is universally Pettis integrable.

(d′) <u>Each</u> x^{**} <u>in</u> X^{**} <u>is universally measurable and satisfies the barycentric formula on</u> (A, weak*) .

<u>Proof</u>. As in the proof of Corollary 5, we may write $A = T^*(B)$ where B denotes the unit ball of $C(A)^*$ and $T : X \longrightarrow C(A)$ is the evaluation mapping. Then each of the statements (a) and (b) is equivalent to the condition that the operator T factors through a Banach space not containing ℓ_1 by Theorem 4 and by Theorem 12 of Saab and Saab [45]. Hence (a) and (b) are equivalent.

Now suppose (a) holds. As mentioned above, T can be factored through a Banach space not containing ℓ_1 . Reread the proof of the implication (b) $\Rightarrow$ (c) in Theorem 4 to see that statement (d) holds. Clearly (d) implies (c).

Towards showing that (c) implies (b), observe that the function $\langle I(\cdot),x\rangle$ is continuous on (A, weak*) for each x in X . Thus the same argument as in the proof of Theorem 3.3 shows that each sequence in the set $\{\langle I(\cdot),x\rangle : ||x|| \leq 1\}$ has a pointwise convergent subsequence. Let (x_n) be a sequence in the unit ball of X and take such a subsequence. Then for each z^* in B,

$$z^*(Tx_n) = T^*z^*(x_n) = \langle I(T^*z^*),x_n\rangle$$

has a convergent subsequence since T^*z^* belongs to $A = T^*(B)$. Therefore the set $T(B_X)$ is weakly precompact, so that $A = T^*(B)$ has the weak Radon-Nikodým property by Theorem 4. This completes the proof.

Corollary 10 : A weak*-compact convex subset of an absolutely convex weak*-compact weak Radon Nikodým set inherits the weak Radon-Nikodým property from the larger set.

Proof. This is an immediate consequence of Theorem 9 and Lemma 3.

Odell and Rosenthal [31] showed that a separable Banach space X does not contain a copy of ℓ_1 if and only if every x^{**} in X^{**} is a limit of a sequence of continuous functions on the unit ball of X^* equipped with the weak*-topology; that is, if and only if every x^{**} is a Baire-1 function on (B_{X^*}, weak^*) . Their result, coupled with that of Janicka [24], showed that the dual X^* of a separable Banach space X has the weak Radon-Nikodým property if and only if every x^{**} in X^{**} satisfies the same Baire-1 criteria. In our next theorem we combine our previous results to yield a localization of these ideas. First, however, we need a lemma that will allow the passage from convex sets to absolutely convex sets.

Lemma 11 : Let X be a separable Banach space and let C_1 and C_2 be two weak*-compact convex subsets of X^* that have the weak Radon-Nikodým property. Then the set $A = C_1 - C_2$ has the weak Radon-Nikodým property.

Proof. The set $A_1 = C_1 \times C_2$ has the weak Radon-Nikodým property in the space $X^* \times X^*$. Let $T : X^* \times X^* \longrightarrow X^*$ be defined by $T(x^*,y^*) = x^* - y^*$. Then T is weak*-to-weak* continuous and $T(A_1) = A$. Let x^{**} be in X^{**} and let μ be a Radon probability measure on (A, weak*) . By [1, page 90], there is a Radon probability measure υ on A_1 such that $T(\upsilon) = \mu$. By Theorem 9, the function $x^{**}T$ is υ-measurable on A_1 , and therefore x^{**} is $T(\upsilon) = \mu$-measurable on A . We have thus shown that the identity map I : (A, weak*) $\longrightarrow X^*$ is universally scalarly measurable. Apply Theorem 9 again to conclude that A has the weak Radon-Nikodým property.

Theorem 12 : *Let* X *be a separable Banach space. Any of the following statements about a weak*-compact convex subset* C *of* X^* *implies all the others.*

(a) *The set* C *has the weak Radon-Nikodým property.*

(b) *The set* C *has the scalar point of continuity property.*

(c) *For every* x^{**} *in* X^{**} , *the restriction of* x^{**} *to* (C, weak*) *is a Baire-1 function.*

(d) For every x^{**} in X^{**} , the restriction of x^{**} to (C, weak*) is universally measurable.

Proof. To see that (a) implies (b), we can suppose that 0 is in C . Let A be the absolutely convex set $C - C$. Since A has the weak Radon-Nikodým property by Lemma 11, it has the scalar point of continuity property by Theorem 9. Therefore $C \subset A$ also has the scalar point of continuity property.

The Baire Characterization Theorem yields the implication (b) $\Rightarrow$ (c) since C is weak*-metrizable. That (c) implies (d) is clear. Now invoke Theorem 8 to finish the proof.

It is clear that (b) does not imply (c) in the above theorem if X is not separable. Let $X = c_o(\Gamma)$ for an uncountable index set Γ. Because ℓ_1 does not embed in X, the unit ball of $X^* = \ell_1(\Gamma)$ has the scalar point of continuity property [45]. On the other hand, suppose $x^{**} \in \ell_\infty(\Gamma)$ is a Baire-1 function when restricted to the unit ball of $\ell_1(\Gamma)$. Then Odell and Rosenthal showed [31] that x^{**} is actually the weak*-limit of a sequence from $c_o(\Gamma)$ and therefore is countably supported. However, there are many elements in $\ell_\infty(\Gamma)$ that are not countably supported. Consequently, statement (c) must fail for the unit ball of $\ell_1(\Gamma)$.

Recently, Elias Saab has shown that statements (a) and (b) in Theorem 12 are equivalent for weak*-compact convex subsets of arbitrary dual spaces. His key step involved using a compactness argument to show that the absolutely convex set $K - K$ has the weak Radon-Nikodým property if the weak*-compact convex set K has the weak Radon-Nikodým property. He was then able to show that the assumption of absolute convexity could be replaced by ordinary convexity in Corollaries 5,6,7 and 10, and in Theorem 9.

§3 The geometry of weak Radon-Nikodým sets

In this section we present several geometric characterizations of weak Radon-Nikodým sets in terms of tree structures, extreme points and a dentability criterion due to Elias and Paulette Saab [45].

We saw in Chapter 2 that dual Banach spaces lacking the weak Radon-Nikodým property provide fertile ground for the growth of Rademacher trees. The next theorem localizes this result.

Theorem 13 : For an operator $T : X \longrightarrow Y$, the set $T^*(B_{Y^*})$ is a weak Radon-Nikodým set if and only if it does not contain a Rademacher tree.

Consequently, a weak*-compact absolutely convex subset of X^* is a weak Radon-Nikodým set if and only if it does not contain a Rademacher tree.

Proof. Suppose $K = T^*(B_{Y^*})$ contains a δ-Rademacher tree (x_n^*) . Define the usual martingale associated with a tree [cf. 10, V.1.7], i.e.,

$$f_1 = x_1^* \chi_{[0,1]}$$

$$f_2 = x_2^* \chi_{[0,\frac{1}{2}]} + x_3^* \chi_{[\frac{1}{2},1]}$$

etc. Define an operator $S : L_1[0,1] \longrightarrow X^*$ by $S(g) = \lim_n \int g f_n \, d\lambda$

(λ is Lebesgue measure on [0,1]) and observe that if r_n is the n^{th} Rademacher function, then

$$||S(r_n)|| = ||\int r_n f_n \, d\lambda|| \geq \delta \ .$$

Since the sequence (r_n) is weakly null in $L_1[0,1]$, we see that the operator S is not Dunford-Pettis. However, the set K does contain $S(\chi_E/\lambda(E))$ since it contains the range of f_n for each n. Invoke Theorem 4 to conclude that $K = T^*(B_{Y^*})$ is not a weak Radon-Nikodym set.

Conversely, suppose $K = T^*(B_{Y^*})$ is not a weak Radon-Nikodým set. Invoke Theorem 4 and Rosenthal's Theorem on weakly precompact sets to find a sequence (Tx_n) in $T(B_X)$ that is a copy of the ℓ_1-basis (e_n). Let Y_o be the closed subspace determined by the sequence (Tx_n). Then $S(Tx_n) = e_n$ defines an isomorphism from Y_o onto ℓ_1. Let (e_n^*) be the c_o-tree described earlier, but now considered as a tree in the sequence space ℓ_∞. Let $y_n^* = ||S||^{-1} S^*(e_n^*)$. Then (y_n^*) is a tree in $B_{Y_o^*}$. By the lifting property of bounded trees in dual spaces (this follows easily from a compactness argument, cf. [6]) there exists a tree (z_n^*) in B_{Y^*} such that $z_n^*(y) = y_n^*(y)$ for all y in Y_o.

We claim that $(T^*z_n^*)$ is a Rademacher tree in K. To see this, first note that for each $k = 1,2,3,\ldots$, letting $i = k + 1$, the element e_i of the ℓ_1-basis satisfies $e_n^*(e_i) = (-1)^n$ for $n = 2^k,\ldots,2^{k+1}-1$. Taking each of the following sums from $n = 2^k$

to $n = 2^{k+1}-1$, we obtain

$$\begin{aligned} || \sum (-1)^n T^*z_n^* || &\geq | \sum (-1)^n T^*z_n^*(x_i) | \\ &= ||S||^{-1} | \sum (-1)^n e_n^*(e_i) | \\ &= ||S||^{-1} | \sum (-1)^n (-1)^n | \\ &= 2^k ||S||^{-1} . \end{aligned}$$

Hence $(T^*z_n^*)$ is a $||S||^{-1}$-Rademacher tree in K .

The second statement follows from the first statement and the observation following Theorem 4. This completes the proof.

A quick glance at the beginning of the above proof shows that any Banach space containing a bounded Rademacher tree fails to have the weak Radon-Nikodým property. Since the isomorphic copy of a Rademacher tree is again a Rademacher tree, this observation immediately shows that neither c_o nor $L_1[0,1]$ can be embedded in a Banach space having the weak Radon-Nikodým property, a fact previously proved by Janicka [24] and by Ghoussoub and Saab [19].

In 1976, Haydon [21] showed that spaces not containing ℓ_1 have fairly strong extremal properties in their duals. Since such duals also have the weak Radon-Nikodým property, the next theorem might be considered as a localization of Haydon's result.

Theorem 14 : Each of the following statements about an operator $T : X \longrightarrow Y$ implies all the others.

(a) The set $T^*(B_{Y^*})$ is a weak Radon-Nikodým set.

(b) Every weak*-compact convex subset of $T^*(B_{Y^*})$ is the norm-closed convex hull of its extreme points.

(c) For every weak*-compact subset W of $T^*(B_{Y^*})$, the weak*-closed convex hull of W coincides with the norm-closed convex hull of W .

Consequently, a weak*-compact absolutely convex subset K of X^* is a weak Radon-Nikodým set if and only if every weak*-compact subset of K is the norm-closed convex hull of its extreme points.

Proof. (a) $\Rightarrow$ (b) . Let $K = T^*(B_{Y^*})$ and let C be a weak*-compact convex subset of K . Suppose $C \neq$ norm-cl conv(Ext C). By the proof of [21, Proposition 3.1], there exists a non-empty subset S of C , a subsequence (x_n) in B_X , and a constant $\delta > 0$ such that

$$\delta \sum |a_i| \leq \sup \{ |\sum a_i s(x_i)| : s \in S\}$$

for all finitely non-zero sequences (a_i) of reals. Fix s in S and observe that $s = T^*y^*$ for some y^* in B_{Y^*} . Then

$$|\sum a_i s(x_i)| = |\sum a_i T^*y^*(x_i)| \leq \|\sum a_i (Tx_i)\| .$$

Accordingly

$$\delta \sum |a_i| \leq \|\sum a_i(Tx_i)\| ,$$

i.e., the sequence (Tx_n) is a copy of the ℓ_1-basis. Consequently $T(B_X)$ is not weakly precompact. Appeal to Theorem 4 to see that $T^*(B_{Y^*})$ is not a weak Radon-Nikodým set.

(b) ⟹ (c) . This follows immediately from the observation that if W is a weak*-compact subset of K , then weak*-cl conv(W) $\subset K$ and ext(weak*-cl conv(W)) $\subset W$ [8, V.1.3].

(c) ⟹ (a) . Suppose $T^*(B_{Y^*})$ is not a weak Radon-Nikodým set. Use Theorem 4 and Rosenthal's Theorem to find a copy (Tx_n) of the ℓ_1-basis in $T(B_X)$. Let X_o denote the closed subspace spanned by the sequence (x_n) and let $i ; X_o \longrightarrow X$ be the natural inclusion map. In addition, let T_o denote the restriction of T to the subspace X_o . Note that since there is a $\delta > 0$ satisfying

$$\delta \|\sum a_i x_i\| \leq \delta \sum |a_i| \leq \|\sum a_i Tx_i\| = \|T_o(\sum a_i x_i)\|$$

for all finitely non-zero sequences (a_i) of reals, the operator T_o has a bounded inverse on its (closed) range Y_o . Therefore its adjoint $T_o^* : Y_o^* \longrightarrow X_o^*$ also has a bounded inverse.

Let $V : Y_o \longrightarrow C[0,1]$ be a quotient map on the separable space Y_o , and let K_o denote the image of the set of unit point masses on [0,1] under the action of the adjoint operator V^* . Then K_o is a weak*-compact subset of Y_o^* that has distinct weak*-closed and norm-closed convex hulls (see [21]). Since weak*-cl conv(K_o)

is weak*-compact and T_o^* is weak*-to-weak* continuous, we have

$$T_o^*(\text{weak*-cl conv}(K_o)) = \text{weak*-cl conv}(T_o^* K_o) .$$

In addition, since T_o^* has a norm continuous inverse,

$$T_o^*(\text{norm-cl conv}(K_o)) = \text{norm-cl conv}(T_o^* K_o) .$$

Therefore norm-cl conv$(T_o^* K_o) \neq$ weak*-cl conv$(T_o^* K_o)$.

Let $j : Y_o \longrightarrow Y$ denote the natural inclusion map. Pick a weak*-compact subset M of Y^* such that $j^*M = K_o$. Let $W = T^*M$ and observe that $i^*W = T_o^* K_o$. Then i^*(weak*-cl conv(W)) = weak*-cl conv$(T_o^* K_o)$ and i^*(norm-cl conv$(W)) \subset$ norm-cl conv$(T_o^* K_o)$, so consequently weak*-cl conv$(W) \neq$ norm-cl conv(W) . A suitable scalar multiple of W produces a weak*-compact subset of $T^*(B_{Y^*})$ with distinct weak*-closed and norm-closed convex hulls, as required.

This completes the proof.

In [45] Saab and Saab introduce the following dentability criterion.

Definition : A bounded subset A of X^* is said to be weak*-scalarly dentable if for every $\alpha > 0$ and every x^{**} in X^{**} there exists a weak*-open slice

$$S = S(A,x,\beta) = \{x^* \in A : x^*(x) > \sup_{y^* \in A} y^*(x) - \beta\}$$

for some x in X and some $\beta > 0$ such that the oscillation of x^{**} on S satisfies

$$\sup \{|x^{**}(x^*) - x^{**}(y^*)| : x^*, y^* \in S\} \leq \alpha .$$

They then proceed to show that every non-empty bounded subset of X^* is weak*-scalarly dentable if and only if for every weak*-compact subset M and every x^{**} in X^{**}, the restriction of x^{**} to M equipped with the weak*-topology has a point of continuity. A close examination of their proof, though, reveals that it works inside any weak*-compact convex set; i.e., for a weak*-compact convex set K, every non-empty subset of K is weak*-scalarly dentable if and only if K has the scalar point of continuity property. By Theorem 9, however, this latter condition characterizes absolutely convex weak*-compact sets with the weak Radon-Nikodým property.

With the help of a standard separation argument, it is easily seen that a non-empty bounded set A is weak*-scalarly dentable if and only if for every $\alpha > 0$ and every x^{**} in X^{**} there exists an x^* in A such that

$$(*) \quad x^* \notin \text{weak*-cl conv}(A \{y^* \in A : |x^{**}(x^*) - x^{**}(y^*)| < \alpha\}) .$$

For suppose this latter condition (*) holds. Then one can find an x in X such that

$$x^*(x) > \xi = \sup \{y^*(x) : y^* \in A \text{ and } |x^{**}(y^*) - x^{**}(x^*)| \geq \alpha\} .$$

Let $\beta = \xi - x^*(x)$ and observe that if y^* belongs to the weak*-open slice $S = S(A,x,\beta)$, then $y^*(x) > \xi$. Therefore

$$\sup \{|x^{**}(y^*) - x^{**}(z^*)| : y^*,z^* \in S\} \leq 2\alpha$$

and hence A is weak*-scalarly dentable.

Conversely, suppose A is weak*-scalarly dentable. Then if $\alpha > 0$ and x^{**} belongs to X^{**} , there exists a weak*-open slice $S = S(A,x,\beta)$ such that

$$\sup \{|x^{**}(x^*) - x^{**}(y^*)| : x^*,y^* \in S\} \leq \alpha/2 .$$

Let x^{**} be in S . Then S is a subset of the weak neighborhood $W = \{y^* \in X^* : |x^{**}(x^*) - x^{**}(y^*)| < \alpha\}$ of x^* . Because S is weak*-open and has convex complement, we obtain

$$A \setminus W \subset A \setminus S$$

$$\Longrightarrow \text{conv}(A \setminus W) \subset \text{conv}(A \setminus S) \subset \text{conv}(A) \setminus S$$

$$\Longrightarrow \text{weak*-cl conv}(A \setminus W) \subset (\text{weak*-cl conv}(A)) \setminus S$$

and therefore we see that x^* does not belong to weak*-cl $\text{conv}(A \setminus W)$.

We obtain as an immediate corollary that a bounded set A

in X^* is weak*-scalarly dentable if and only if its weak*-closed convex hull is weak*-scalarly dentable. To see this just mimic the proof for the ordinary dentable case (see, for example, [10, V.10(i)]).

It is worth remarking here what happens if various parts of this dentability condition are changed. If, for example, one takes the weak closure (or equivalently the norm closure) in condition (*), then the Bishop-Phelps Theorem insures that every non-empty subset of X^* satisfies the new condition (as was pointed out to us by Elias Saab). This immediately shows that weakly compact sets in dual spaces are weak*-scalarly dentable, a result that is not surprising since such sets are actually dentable [10, V.10(ii)]. Likewise, if one requires the conditions in the definition to hold only for every x in X rather than for all elements in X^{**}, then again all non-empty subsets have the corresponding property. Finally, if one requires that the norm diameter of the weak*-open slice be less that α, then the property is called weak*-dentability and every non-empty bounded set has the property if and only if X^* has the Radon-Nikodým property (see Namioka and Phelps [30]).

Recall that a point x^* in a subset C of X^* is called a <u>weak*-strongly exposed point</u> of C if there exists an x in X of norm one such that

$$x^*(x) = \sup \{y^*(x) : y^* \in C\}$$

and x_n^* converges to x^* in norm whenever $x_n^*(x)$ converges to $x^*(x)$

for sequences (x_n^*) in C. Weak*-strongly exposed points play an important role in the geometry of Banach spaces. It follows from Namioka and Phelps [30], for example, that a dual space X^* has the Radon-Nikodým property if and only if each weak*-compact convex non-empty subset of X^* contains a weak*-strongly exposed point. A set containing such an exposed point is also weak*-dentable.

Let us say that a point x^* in a subset C of X^* is a <u>weak*-weakly exposed point</u> of C if there exists an x in X of norm one such that

$$x^*(x) = \sup \{y^*(x) : y^* \in C\}$$

and x_n^* converges weakly to x^* whenever $x_n^*(x)$ converges to $x^*(x)$ for sequences (x_n^*) in C. Do weak*-weakly exposed points play an analogus role for the weak Radon-Nikodým property in dual spaces as weak*-strongly exposed points play for the Radon-Nikodým property? It is easy to see, for example, that if every weak*-compact convex subset of X^* has a weak*-weakly exposed point x^*, then X^* has the scalar point of continuity property since every x^{**} in X^{**} is weak*-continuous at x^*. As we have seen earlier, this implies that X^* has the weak Radon-Nikodým property. Does the converse hold or does the existence of weak*-weakly exposed points actually give us a stronger property (for example, the Radon-Nikodým property)?

The last theorem summarizes the discussion of this chapter. The interested reader should keep in mind the remarks in the Introduction when comparing this theorem with Theorem 1.

Theorem 16 : Each of the following statements about an absolutely convex weak*-compact subset K of X^* implies all the others.

(a) Every x^{**} in X^{**} is universally measurable and satisfies the barycentric formula on (K, weak^*).

(b) Every bounded linear operator $T : L_1(\mu) \longrightarrow X^*$ satisfying $T(\chi_E/\mu(E)) \in K$ for $\mu(E) > 0$ is a Dunford-Pettis operator.

(c) The set K has the Radon-Nikodým property for the Pettis integral.

(d) The restriction of each x^{**} in X^{**} to each non-empty weak*-compact subset of K has a point of weak*-continuity.

(e) The set K does not contain a Rademacher tree.

(f) Every weak*-compact convex subset of K is the norm-closed convex hull of its extreme points.

(g) Every non-empty bounded subset of K is weak*-scalarly dentable.

CHAPTER 5

The Bourgain property and applications

§1 The Bourgain property

In the previous chapters we have seen that the family $\{<f(\cdot),x> : ||x|| \leq 1\}$ plays a strong role in determining Pettis integrability for a bounded scalarly measurable function f from Ω into a dual space X^*. We continue this approach in this chapter, but, rather than viewing such families as subsets of $L_\infty(\mu)$, we now consider them simply as families of real-valued functions on Ω. A property of real-valued functions formulated by J. Bourgain [3] is the cornerstone of our discussion.

Definition : Let (Ω,Σ,μ) be a measure space. A family Ψ of real-valued functions on Ω is said to have the Bourgain property if the following condition is satisfied: for each set A of positive measure and for each $\alpha > 0$, there is a finite collection F of subsets of A of positive measure such that for each function f in Ψ, the inequality $\sup f(B) - \inf f(B) < \alpha$ holds for some member B of F.

Let $f : \Omega \longrightarrow X^*$ be a bounded scalarly measurable function. Fix x^{**} in X^{**} and use Goldstine's Theorem to find a bounded net (x_β) in X that converges to x^{**} in the weak*-topology. Let x_E^* be the Gel'fand integral of f over a set E in Σ and note that

$$x^{**}(x_E^*) = \lim_\beta x_E^*(x_\beta) = \lim_\beta \int_E \langle f, x_\beta \rangle \, d\mu \, .$$

Now if we could take the last limit underneath the integral sign, then we would have

$$x^{**}(x_E^*) = \int_E \lim_\beta \langle f, x_\beta \rangle \, d\mu = \int_E x^{**}f \, d\mu \, ,$$

and this would prove that f is Pettis integrable. Unfortunately, it is not always possible to take the limit underneath the integral sign but it is always possible to do so if the net (x_β) can be replaced by a sequence. The next theorem, which is due to Bourgain [3], essentially allows us to do this for some functions f .

Theorem 1 : If (Ω, Σ, μ) is a finite measure space and Ψ is a family of real-valued functions on Ω satisfying the Bourgain property, then

(i) the pointwise closure of Ψ satisfies the Bourgain property;

(ii) each element in the pointwise closure of Ψ is measurable, and

(iii) each element in the pointwise closure of Ψ is the almost everywhere pointwise limit of a sequence from Ψ .

Proof. The proof of (i) is completely straightfoward. Towards verifying (ii) and (iii), take a function g belonging to the pointwise closure of Ψ and an ultrafilter U on Ψ that has g as a cluster point. For A in Σ and $\alpha > 0$, let

$$\Psi(A;\alpha) = \{f \in \Psi : \sup f(A) - \inf f(A) < \alpha\} .$$

It follows from the definition of the Bourgain property that if A has positive measure and $\alpha > 0$, then there exists a subset B of A of positive measure with $\Psi(B;\alpha)$ belonging to U. Now for each $\alpha > 0$, use Zorn's Lemma to find a maximal set P_α of mutually disjoint sets of positive measure such that $\Psi(A;\alpha) \in U$ for each $A \in P_\alpha$. Note that each P_α is necessarily countable. Moreover,

(a) the set $\Omega \setminus \cup P_\alpha$ has measure 0 for each $\alpha > 0$,

and

(b) if F is a finite subset of positive reals and Q_α is a finite subset of P_α for each α in F, then g belongs to the pointwise closure of $\bigcap_{\alpha \in F} \bigcap_{A \in Q_\alpha} \Psi(A;\alpha)$.

The maximality of P_α yields condition (a), and condition (b) follows because g is a cluster point of U.

Now let $(A_{m,n})_n$ be an enumeration of $P_{1/m}$, and set

$$B = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} A_{m,n} .$$

By condition (a), we have $\mu(\Omega \setminus B) = 0$. Pick some point $\omega_{m,n}$ in each set $A_{m,n}$ and define

$$f_m = \sum_{n=1}^{\infty} g(\omega_{m,n})\chi_{A_{m,n}} .$$

Each f_m is measurable and a quick computation using (b) shows that the sequence (f_m) converges to g uniformly on B . Therefore g is measurable.

Unfortunately, the functions f_m may not belong to Ψ . To establish (iii), therefore, use condition (b) to pick for each integer m a function h_m belonging to $\bigcap_{i=1}^{m} \bigcap_{n=1}^{m} \Psi(A_{i,n};1/i)$ such that

$$|h_m(\omega_{i,n}) - g(\omega_{i,n})| < 1/i$$

for each $1 \leq i,n \leq m$. The triangle inequality now ensures that $(h_m(\omega))$ converges to $g(\omega)$ for each ω in B . This completes the proof.

It is worth remarking here that a uniformly bounded family Ψ of real-valued functions has the Bourgain property if and only if the following condition holds:

(*) for each non-null measurable set A in Σ and for each pair $a < b$ of real numbers, there is a finite collection F of non-null measurable subsets of A such that for each f in Ψ , either $\inf f(B) \geq a$ or $\sup f(B) \leq b$ for some member B of F .

Indeed, the Bourgain property for Ψ with $\alpha = b-a$ clearly implies property (*); conversely, the Bourgain property for Ψ can be obtained by finitely many successive applications of property (*).

§2 Pettis integration and the Bourgain property

In this section we study the family $\{\langle f,x\rangle : ||x|| \leq 1\}$ for a bounded function $f : \Omega \longrightarrow X^*$ and use the Bourgain property to determine the Pettis integrability of the function. We shall say that f has the Bourgain property if the family $\{\langle f,x\rangle : ||x|| \leq 1\}$ has the Bourgain property. The main result is the following sufficient condition.

Theorem 2 : A bounded function $f : \Omega \longrightarrow X^*$ that has the Bourgain property is Pettis integrable.

Proof. While no *a priori* hypothesis about the measurability of f is assumed, the Bourgain condition does show immediately that $\langle f,x\rangle$ is measurable for each x in X. Fix x^{**} in the unit ball of X^{**} and fix a set E in Σ. Let x_E^* be the Gel'fand integral of f over E, so that

$$(1) \qquad x_E^*(x) = \int_E \langle f,x\rangle \, d\mu \quad \text{for all} \quad x \in X .$$

We must show that $x^{**}f$ is measurable and that

$$x^{**}(x_E^*) = \int_E x^{**}f \, d\mu \, .$$

Accordingly, let $\alpha > 0$ and define

$$\Psi = \{\langle f,x\rangle : ||x|| \leq 1 \, , \, |\langle x^{**} - x, x_E^*\rangle| < \alpha\} \, .$$

Goldstine's theorem ensures that $x^{**}f$ lies in the pointwise closure of Ψ. Since Ψ has the Bourgain property, the function $x^{**}f$ is measurable by Theorem 1(ii), and statement (iii) of the same theorem shows that $x^{**}f$ is the almost everywhere limit of a sequence $\langle f,x_n\rangle$ from Ψ; that is,

(2) $$\lim_n \langle f,x_n\rangle = x^{**}f \quad \text{a.e.} \, , \text{ where}$$

(3) $$|x^{**}(x_E^*) - x_E^*(x_n)| < \alpha \quad \text{for each } n \, .$$

It now follows from equations (1), (2), (3) and the Dominated Convergence Theorem that

$$\left|x^{**}(x_E^*) - \int_E x^{**}f \, d\mu\right| < \alpha \, .$$

Since α was arbitrary, we conclude that x_E^* is the Pettis integral of f over the set E.

In Chapter 3 we saw that a bounded function f taking values in the dual of a separable space is Pettis integrable if the family $\{\langle f,x\rangle : ||x|| \leq 1\}$ is almost weakly precompact in $L_\infty(\mu)$. A natural question arises concerning how Theorem 2 relates to this earlier result. Before answering this, we first establish some additional facts about the family $\{\langle f,x\rangle : ||x|| \leq 1\}$ and the Bourgain property.

Lemma 3 : Suppose $f : \Omega \longrightarrow X^*$ and $g : \Omega \longrightarrow X^*$ are equal almost everywhere. Then f has the Bourgain property if and only if g has the Bourgain property.

Proof. Let N be a null set such that $f(\omega) = g(\omega)$ for all ω not in N. Clearly both $\langle f,x\rangle$ and $\langle g,x\rangle$ have the same supremum and infimum on the set $A\setminus N$ for any set A of positive measure. The conclusion now follows immediately.

For the rest of this chapter we shall assume that (Ω,Σ,μ) is a finite separable measure space. This means that there is a sequence (π_n) of finite partitions of Ω consisting of sets in Σ satisfying:

(1) each member of π_{n+1} is contained in a member of π_n (i.e., π_{n+1} refines π_n), and

(2) the union of the σ-algebras generated by the partitions π_n is dense in Σ.

For example, if $\Omega = [0,1]$ and μ is Lebesgue measure on the Borel sets Σ, then the dyadic partitions of $[0,1]$ would satisfy these assumptions. Let Σ_n denote the σ-algebra generated by π_n and let $\sigma = \bigcup_{n=1}^{\infty} \Sigma_n$.

Lemma 4 (Bourgain) : *Suppose* A *is a subset of* Ω *with positive measure and* $0 < \alpha < 1$. *Then there is an integer* m *and a measurable subset* $B \subset A$ *with* $\mu(B) > (1-\alpha)\mu(A)$ *such that for every uniformly bounded by 1 real-valued martingale* (g_n, Σ_n) *and for every* $n \geq m$,

(i) $\text{ess inf } g(A) \leq \inf g_n(B) + \alpha$

(ii) $\text{ess sup } g(A) \geq \sup g_n(B) - \alpha$

where g *is any almost everywhere limit of the sequence* (g_n) .

Proof. Choose $a,b > 0$ so that $1 - \alpha/4 < a < 1$, $b < 1$, and $1 + a < 2b$. Choose an integer m and a set A_1 in Σ_m such that $\mu(A \,\Delta\, A_1) < (1-b)^2\mu(A)$. Now let

$$\Pi = \{E \in \sigma : \mu(E \cap A_1 \setminus A) > (1-a)\mu(E)\}$$

and set $W = \cup\, \Pi$. By writing W as a countable union of disjoint sets from Π , we can easily see that $\mu(W \cap A_1 \setminus A) > (1-a)\mu(W)$. If we let $C = \Omega \setminus W$, then

$$\mu(C) = 1 - \mu(W) > 1 - \frac{\mu(A_1 \; A)}{1-a} > 1 - (1-b)\mu(A)$$

and

$$\mu(E \cap A_1 \setminus A) \leq (1-a)\mu(E)$$

whenever E is in σ and $E \cap C \neq \emptyset$.

We claim that the integer m and the set $B = A \cap A_1 \cap C$ satisfy the stated conditions. First of all, notice that

$$\mu(B) \geq \mu(A \cap A_1) - \mu([0,1]\setminus C) \geq \mu(A \cap A_1) - (1-b)\mu(A)$$

$$> \mu(A) - (1-b)^2\mu(A) - (1-b)\mu(A) > a\mu(A)$$

$$> (1-\alpha)\mu(A) .$$

We next verify condition (i) (the argument for (ii) follows by replacing g_n with $-g_n$). Suppose $n \geq m$ and β is any number satisfying $\inf g_n(B) < \beta < 1 + \alpha$. Since g_n is constant on the members of π_n , there is some interval I in π_n such that $I \cap B$ is non-empty and $g_n < \beta$ on I . Moreover, we have $I \subset A_1$ since $I \cap A_1$ is non-empty and A_1 is a union of sets in π_n . But because $I \cap C \neq \emptyset$, we see that

$$\mu(I \cap A) = \mu(I) - \mu(I \cap A_1 \setminus A) \geq a\mu(I) .$$

Now suppose $\operatorname{ess\,inf} g(A) > \beta + \alpha$. Because g_n is the conditional expectation of g with respect to the σ-algebra Σ_n , we have

$$\beta\mu(I) > \int_I g_n \, d\mu = \int_I g \, d\mu = \int_{I\cap A} g \, d\mu + \int_{I\setminus A} g \, d\mu$$

$$> (\beta+\alpha)\mu(I \cap A) - \mu(I \setminus A) = (\beta+\alpha+1)\mu(I \cap A) - \mu(I)$$

$$\geq (\beta+\alpha+1)a\mu(I) - \mu(I) .$$

Hence $(\beta+\alpha+1)a - 1 < \beta$, and this implies the contradiction $\beta > 3 - \alpha$. Therefore ess inf $g(A) \leq \beta + \alpha$, and the proof is complete.

Let $f : \Omega \longrightarrow X^*$ be a bounded weak*-scalarly measurable function and define an X^*-valued martingale (f_n, Σ_n) by

$$f_n(\cdot) = \sum_{A \in \pi_n} \frac{G\text{-}\int_A f \, d\mu}{\mu(A)} \chi_A(\cdot)$$

where $G\text{-}\int_A f \, d\mu$ is the Gel'fand integral of f over the set A . Without loss of generality we may assume that $||f|| \leq 1$ pointwise. Then for each x in X , the sequence $(\langle f_n, x\rangle, \Sigma_n)$ is a uniformly bounded by 1 real-valued martingale with

$$\lim_n \langle f_n, x\rangle = \langle f, x\rangle \quad \text{a.e.} ,$$

where the exceptional null set may, of course, vary with x .

Lemma 5 : Let X be a separable Banach space. Then f has the Bourgain property if and only if the family $\{\langle f_n, x\rangle : n \in N , ||x|| \leq 1\}$ has the Bourgain property.

Proof. Let (x_m) be a dense sequence in X . For each integer m there exists a null set N_m satisfying

$$\lim_n \langle f_n(\omega), x_m\rangle = \langle f(\omega), x_m\rangle$$

for each ω not in N_m . Because the sequence (x_n) is dense, it follows easily that for each x in X ,

$$\lim_n \langle f_n(\omega), x\rangle = \langle f(\omega), x\rangle$$

for each ω that is not in the null set $N = \bigcup_{m=1}^{\infty} N_m$.

Suppose first that the family $\{\langle f_n, x\rangle : n \in N \, , \, ||x|| \leq 1\}$ has the Bourgain property. When the ball of X^* is equipped with the weak*-topology, the space of functions from Ω into B_{X^*} is compact for the topology of pointwise convergence. Therefore, there exists a pointwise weak*-cluster point $g : \Omega \longrightarrow X^*$ of the sequence (f_n) . This means that the family $\{\langle g, x\rangle : ||x|| \leq 1\}$ belongs to the pointwise closure of the family $\{\langle f_n, x\rangle : n \in N \, , \, ||x|| \leq 1\}$. Consequently, the function g has the Bourgain property by Theorem 1(i). A moment's reflection, however, shows that $\langle f(\omega), x\rangle = \langle g(\omega), x\rangle$ for each ω not in N and for each x in X . Hence $f = g$ almost everywhere. Now invoke Lemma 3 to see that f has the Bourgain property.

Conversely, suppose that the family $\{\langle f, x\rangle : ||x|| \leq 1\}$ has the Bourgain property. Let A be a set of positive measure and let $a < b$. Choose $\alpha > 0$ such that $a + \alpha < b - \alpha$. There exist non-null subsets $A_1, \ldots, A_k$ of A such that for each x in the ball of X, either $\sup_{A_i} \langle f,x\rangle \le b - \alpha$ or $\inf_{A_i} \langle f,x\rangle \ge a + \alpha$ for some i.

According to Lemma 4, there is for each set A_i an integer m_i and a non-null subset B_i of A_i such that

$$\text{(a)} \quad \text{ess}\inf_{A_i} \langle f,x\rangle \le \inf_{B_i} \langle f_n,x\rangle + \alpha$$

$$\text{(b)} \quad \text{ess}\sup_{A_i} \langle f,x\rangle \ge \sup_{B_i} \langle f_n,x\rangle - \alpha$$

for every x in the ball of X and for every $n \ge m_i$. Let $m = \max\{m_i : 1 \le i < k\}$. Let $n \ge m$, let x be in the ball of X, and note that there exists an A_i such that either

$$b - \alpha \ge \sup_{A_i} \langle f,x\rangle \ge \text{ess}\sup_{A_i} \langle f,x\rangle$$
$$\ge \sup_{B_i} \langle f_n,x\rangle - \alpha$$

or

$$a + \alpha \le \inf_{A_i} \langle f,x\rangle \le \text{ess}\inf_{A_i} \langle f,x\rangle$$
$$\le \inf_{B_i} \langle f_n,x\rangle + \alpha .$$

That is, either $b \ge \sup_{B_i} \langle f_n,x\rangle$ or $a \le \inf_{B_i} \langle f_n,x\rangle$. Therefore the sets $B_1, \ldots, B_k$ will work for the set A for the family $\{\langle f_n,x\rangle : n \ge m, \|x\| \le 1\}$. However, the functions $f_1, \ldots, f_{m-1}$

are just simple functions, so that for each $i = 1,\ldots,m-1$ there exists a set C_i on which f_i is constant and $\mu(A \cap C_i) > 0$. Thus the sets $B_1,\ldots,B_k,C_1 \cap A,\ldots,C_{m-1} \cap A$ will work to show that the family $\{\langle f_n,x\rangle : n \in N , ||x|| \leq 1\}$ has the Bourgain family.

We are now ready to answer the question raised earlier by showing that the Bourgain property for a function f falls between almost weak precompactness in $L_\infty(\mu)$ and Pettis integrability, in the sense of the following theorem.

Theorem 6 : Let X be a separable Banach space and let $f : \Omega \longrightarrow X^*$ be a bounded weak*-scalarly measurable function. If the family $\{\langle f,x\rangle : ||x|| \leq 1\}$ is almost weakly precompact in $L_\infty(\mu)$, then f has the Bourgain property, and hence f is Pettis integrable.

Proof. Observe first that the family $\{\langle f,x\rangle : ||x|| \leq 1\}$ has the Bourgain property if and only if for each $\alpha > 0$ there exists a set E in Σ with $\mu(\Omega \setminus E) < \alpha$ such that the family $\{\langle f,x\rangle \chi_E : ||x|| \leq 1\}$ has the Bourgain property. To see this, take a set A in Σ with $\mu(A) = \alpha > 0$ and apply the Bourgain condition to the non-null set $E \cap A$, where E satisfies the above hypothesis. Without loss of generality, therefore, we may delete the "almost" and assume that $\{\langle f,x\rangle : ||x|| \leq 1\}$ is weakly precompact in $L_\infty(\mu)$. We will also assume that $||f|| \leq 1$.

By Lemma 5 it suffices to show that the family $\{\langle f_n,x\rangle :$

$n \in N$, $||x|| \leq 1\}$ has the Bourgain property. Suppose it does not. Then an argument due to Bourgain [3] produces a sequence (x_n) in the ball of X , a system $(A_{n,m}), n \in N, 1 \leq m \leq 2^n$, of sets of positive measure, and constants $\delta < \beta$ satisfying:

(1) $A_{n+1,2m-1} \subset A_{n,m}$ and $A_{n+1,2m} \subset A_{n,m}$;

(2) $<f(\omega), x_{n+1}> < \delta$ if $\omega \in A_{n+1,2m-1}$;

(3) $<f(\omega), x_{n+1}> > \beta$ if $\omega \in A_{n+1,2m}$.

We sketch the inductive step in the construction. Let $A \in \Sigma$ and $a < b$ be reals for which property (*) (page 71) can not be obtained. For each $m=1,\ldots,2^n$ Lemma 4 provides an integer k_m and a subset $B_m \subset A_{n,m}$ of positive measure such that for $k \geq k_m$ and x in the ball of X ,

$$\text{ess inf}_{A_{n,m}} <f,x> \leq \inf_{B_m} <f_k,x> + \alpha$$

$$\text{ess sup}_{A_{n,m}} <f,x> \geq \sup_{B_m} <f_k,x> - \alpha$$

where $\alpha > 0$ has been chosen so that $a + \alpha < b - \alpha$. Set $j = \max\{k_m : 1 \leq m < 2^n\}$ and for each $m=1,\ldots,2^n$ chose a subset C_m of B_m that has positive measure and is contained in a member of the partition π_j . The negation of the Bourgain property produces some integer k and x_{n+1} in the ball of X such that

$\inf_{C_m} \langle f_k, x_{n+1} \rangle < a$ and $\sup_{C_m} \langle f_k, x_{n+1} \rangle > b$ for all $m=1,\dots,2^n$. Since f_k is constant on each member of π_k , it is clear that $k > j$ and therefore

$$\text{ess inf}_{A_{n,m}} \langle f, x_{n+1} \rangle \le \inf_{B_m} \langle f_k, x_{n+1} \rangle + \alpha < a + \alpha = \delta$$

$$\text{ess sup}_{A_{n,m}} \langle f, x_{n+1} \rangle \ge \sup_{B_m} \langle f_k, x_{n+1} \rangle - \alpha > b - \alpha = \beta$$

for each $m=1,\dots,2^n$. Consequently, the sets $A_{n+1,2m-1} = \{\omega \in A_{n,m} : \langle f(\omega), x_{n+1} \rangle < \delta\}$ and $A_{n+1,2m} = \{\omega \in A_{n,m} : \langle f(\omega), x_{n+1} \rangle > \beta\}$ have positive measure.

Let $O_n = \bigcup_{m=1}^{2^{n-1}} A_{n,2m-1}$ and $E_n = \bigcup_{m=1}^{2^{n-1}} A_{n,2m}$ for each integer n . Then the sequence of pairs (O_n, E_n) is independent in the sense of Rosenthal [42]. More, however, is true in this case, for we actually have

$$\left(\bigcap_{n \in G} O_n \cap \bigcap_{n \in B} E_n \right) \setminus N \neq \emptyset$$

for every pair of disjoint finite non-empty subsets G and B of integers and for every null set N . Rosenthal's argument (see [42] or see page 29) therefore shows that the sequence $(\langle f, x_n \rangle)$ is a copy of the ℓ_1-basis in the $L_\infty(\mu)$-norm, rather than in just the supremum norm. Since this contradicts the assumption that the family $\{\langle f, x \rangle : ||x|| \le 1\}$ is weakly precompact in $L_\infty(\mu)$, we conclude that the family $\{\langle f_n, x \rangle : n \in N , ||x|| \le 1\}$ has the Bourgain property.

It is impossible to dispense entirely with the separability assumption in Theorem 6. As we have seen earlier, the family $\{<f,x> : ||x|| \leq 1\}$ associated with Phillips's function from $[0,1]$ into $\ell_\infty[0,1]$ is actually relatively norm compact, and hence weakly precompact in $L_\infty(\mu)$, but can not have the Bourgain property since f is not Pettis integrable. We can exhibit this directly, however, without invoking Theorem 2.

Example 7 : Under the assumption of the continuum hypothesis, Sierpinski constructed a subset B of the unit square $[0,1] \times [0,1]$ with the properties:

(1) for each t_o in $[0,1]$, the set $\{s : (s,t_o) \in B\}$ is countable;

(2) for each s_o in $[0,1]$, the set $\{t : (s_o,t) \notin B\}$ is countable.

Phillips's function $f : [0,1] \longrightarrow \ell_\infty[0,1]$ is defined [34] by $f(s)(\cdot) = \chi_B(s,\cdot)$ for each s in $[0,1]$. We shall show that the family $\{<f,x> : x \in \ell_1[0,1]\ ,\ ||x|| \leq 1\}$ fails the Bourgain property.

Let $D_1,\ldots,D_n$ be arbitrary non-null subsets of $[0,1]$. For each $i=1,\ldots,n$ choose a point s_i in D_i and note that the set

$$K = \bigcup_{i=1}^{n} \{t : (s_i,t) \notin B\}$$

is countable. Hence we can choose distinct numbers $t_j \in D_j \backslash K$ for each $j=1,\ldots,n$. Thus $(s_i,t_j) \in B$ for all $1 \leq i,j \leq n$. Define an element x in the unit ball of $\ell_1[0,1]$ by $x(t_j) = 1/n$ for $j=1,\dots,n$ and $x(t) = 0$ otherwise. For each $i=1,\dots,n$, observe that

$$\langle f(s_i),x\rangle = \sum_{j=1}^{n} \chi_B(s_i,t_j)x(t_j) = 1 .$$

This shows that

$$\sup_{D_i} \langle f,x\rangle = 1 \quad \text{for} \quad i=1,\dots,n .$$

On the other hand, the set $A = \bigcup_{j=1}^{n} \{s : (s,t_j) \in B\}$ is also countable. Now choose s_i* in $D_i \setminus A$ for each $i=1,\dots,n$ to see that

$$\langle f(s_i*),x\rangle = \sum_{j=1}^{n} \chi_B(s_i*,t_j)x(t_j) = 0 ;$$

that is,

$$\inf_{D_i} \langle f,x\rangle = 0 \quad \text{for} \quad i=1,\dots,n .$$

Therefore the Bourgain property fails for this function.

To conclude this section we present several examples that lend credibility to the conjecture that for bounded dual-valued functions, the Bourgain property is equivalent to Pettis integrability. For this to be true, of course, it is necessary that f have the Bourgain property if and only if g has the Bourgain property whenever

$x^{**}f = x^{**}g$ almost everywhere for all x^{**} in X^{**} (where the exceptional set may depend on x^{**}). Lemma 3 ensures that this holds at least when X is separable.

Example 8 : All strongly measurable functions into X^* have the Bourgain property. In particular, all Bochner integrable functions in X^* have the Bourgain property.

To see this, suppose $f : \Omega \longrightarrow X^*$ is strongly measurable and let (s_n) be a sequence of simple functions for which

$$\lim_n ||f - s_n|| = 0 \quad \text{a.e.}$$

Let A be a measurable subset of Ω with $\mu(A) > 0$, and let $\alpha > 0$. Egorov's theorem ensures the existence of a set B with $\mu(\Omega\setminus B) < \mu(A)$ such that the sequence (s_n) converges uniformly to f on B . Choose an integer n so that $||f(\omega) - s_n(\omega)|| < \alpha/4$ for all ω in B . Since $\mu(A \cap B) > 0$ we can find a set C on which s_n is constant and for which $\mu(A \cap B \cap C) > 0$. Let x be in the ball of X . Then for all ω_1,ω_2 in $A \cap B \cap C$, the triangle inequality shows that

$$|\langle f(\omega_1),x\rangle - \langle f(\omega_2),x\rangle| \leq \alpha/4 + 0 + \alpha/4 = \alpha/2 \ .$$

Therefore

$$\sup_{A\cap B\cap C}\langle f,x\rangle - \inf_{A\cap B\cap C}\langle f,x\rangle < \alpha$$

for all x in the ball of X .

The next three examples present well-known bounded universally Pettis integrable functions on $[0,1]$. In each case we shall show that they satisfy the Bourgain property with respect to any Radon probability measure on $[0,1]$. We shall need the following technical lemma.

Lemma 9 : Let μ be a Radon probability measure on $[0,1]$ and let A be a Borel subset of $[0,1]$ with $\mu(A) > 0$. If $\mu(\{t\})=0$ for each t in A , then for any integer p there is a dyadic partition of $[0,1]$ containing at least p intervals $I_1,\ldots,I_p$ such that $\mu(A \cap I_i) > 0$ for $i=1,\ldots,p$.

Proof. The result clearly holds for $p = 1$. To finish the proof by induction on p , it suffices to prove the result for $p = 2$ for a set of the form $A \cap I$, where I is a dyadic interval and $\mu(A \cap I) > 0$. Towards this end, bisect I into two equal subintervals I_1 and I_2 . If $\mu(A \cap I_1) > 0$ and $\mu(A \cap I_2) > 0$, then we are done. If not, then for one of the intervals, call it E_1 , we must have $\mu(A \cap E_1) = \mu(A \cap I)$. Now bisect E_1 into equal subintervals. Again, either A intersects both intervals in positive measure, in which case we are done, or for one of the intervals, call it E_2 , we have $\mu(A \cap E_2) = \mu(A \cap E_1) = \mu(A \cap I)$.

If the bisecting does not stop, continue in this way to produce a decreasing sequence (E_n) of intervals whose lengths go to 0 and that satisfy $\mu(A \cap E_n) = \mu(A \cap I)$ for each integer n. The first condition, however, implies that $\bigcap_{n=1}^{\infty} (A \cap E_n)$ is either empty or a singleton. Hence $\mu(A \cap E_n)$ converges to 0, which clearly contradicts the second condition. Therefore, at some stage we must bisect E_n into two dyadic intervals that intersect A in sets of positive measure, thereby concluding the proof.

Example 10 : Consider Hagler's function from Example 3.4 given by

$$f(t) = (\chi_{A_n}(t)) \in \ell_\infty$$

for each t in $[0,1]$. In that example we showed that f is universally Pettis integrable on $[0,1]$.

Let μ be a Radon probability measure on $[0,1]$. Suppose A is a non-null subset of $[0,1]$ and $a < b$ are real numbers. If $\mu(\{t\}) > 0$ for some t in A, then for the set $B = \{t\}$ and for every x in the ball of ℓ_1, either $\sup_B \langle f,x \rangle \leq b$ or $\inf_B \langle f,x \rangle \geq a$. We may therefore assume that $\mu(\{t\}) = 0$ for every t in A.

Choose an integer $p > 1/(b-a)$. By Lemma 9 there is a dyadic partition π_m and there are p distinct intervals $I_1, \ldots, I_p$ in π_m with $\mu(A \cap I_i) > 0$ for each $i=1,\ldots,p$. Suppose there exists an $x = (\xi_n)$ in the ball of ℓ_1 such that

$$\sup_{A\cap I_i} \langle f,x\rangle > b \ ,$$

$$\inf_{A\cap I_i} \langle f,x\rangle < a \ ,$$

for each $i=1,\dots,p$. Fix an i between 1 and p . There are t_1^i , t_2^i in $A \cap I_i$ such that $\langle f(t_1^i),x\rangle > b$ and $\langle f(t_2^i),x\rangle < a$. Let

$$M_i = \{n : A_n \in \pi_k \text{ for some } k \geq m \text{ and } t_1^i \in A_n\} ,$$

$$N_i = \{n : A_n \in \pi_k \text{ for some } k \geq m \text{ and } t_2^i \in A_n\} .$$

Next observe that

$$\langle f(t_1^i),x\rangle = \beta + \sum_{n\in M_i} \xi_n ,$$

$$\langle f(t_2^i),x\rangle = \beta + \sum_{n\in N_i} \xi_n ,$$

where β is the sum of all ξ_n for which both t_1^i and t_2^i lie in $A_n \in \pi_k$ for $k < m$. Therefore

$$\sum_{n\in M_i} \xi_n - \sum_{n\in N_i} \xi_n > b - a .$$

Because an interval in π_k for $k \geq m$ can intersect at most one I_i in π_m , we see that $M_i \cap M_j = \emptyset$ and $N_i \cap N_j = \emptyset$ for distinct i and j . Consequently,

$$p(b-a) < \sum_{i=1}^{p} \Big(\sum_{n\in M_i} \xi_n - \sum_{n\in N_i} \xi_n \Big)$$

$$\le \sum_{n=1}^{\infty} |\xi_n| \le 1 < p(b-a) .$$

This contradiction shows that for each x in the ball of ℓ_1, either $\sup_{A\cap I_i} \langle f,x\rangle \le b$ or $\inf_{A\cap I_i} \langle f,x\rangle \ge a$ for some i. Therefore the family $\{\langle f,x\rangle : ||x|| \le 1\}$ has the Bourgain property with respect to μ.

Example 11 : Consider the function from Example 3.5 given by

$$f(t) = \chi_{[0,t]} \in L_\infty[0,1]$$

for each t in $[0,1]$. In that example we showed that f is universally Pettis integrable on $[0,1]$.

Let μ be any Radon probability measure on $[0,1]$, let A be a measurable subset of $[0,1]$ with $\mu(A) > 0$, and let $a < b$ be real numbers. As in Example 10, we may assume that $\mu(\{t\}) = 0$ for each t in A.

Choose an integer n such that $n(b-a) > 1$ and by Lemma 9 choose a dyadic partition π_m such that there are n distinct intervals $I_1,\ldots,I_n$ in π_m with $\mu(A \cap I_i) > 0$ for each $i=1,\ldots,n$. Suppose there exists an x in the ball of $L_1[0,1]$ with

$$\sup_{A\cap I_i} \langle f,x\rangle > b ,$$

$$\inf_{A\cap I_i} \langle f,x\rangle < a ,$$

for each $i=1,\ldots,n$. Then for each $i=1,\ldots,n$ there are points s_i and t_i in $A \cap I_i$ such that

$$\int_0^{s_i} x \, d\mu > b$$

$$\int_0^{t_i} x \, d\mu < a .$$

Without loss of generality assume $s_i > t_i$ (if not, just relabel the two points). Then

$$\int_{t_i}^{s_i} |x| \, d\mu > b - a$$

for each $i=1,\ldots,n$, so that

$$n(b-a) > 1 \geq \int_0^1 |x| \, d\mu \geq \sum_{i=1}^{n} \int_{t_i}^{s_i} |x| \, d\mu > n(b-a) .$$

Since this last inequality is blatantly false, it must happen that either $\sup_{A\cap I_i} \langle f,x\rangle \leq b$ or $\inf_{A\cap I_i} \langle f,x\rangle \geq a$ for some i . This shows that f has the Bourgain property with respect to μ .

Example 12 : Let $\{e_t : t \varepsilon [0,1]\}$ be an orthonormal basis for the nonseparable Hilbert space $\ell_2[0,1]$. Define a function $f : [0,1] \longrightarrow \ell_2[0,1]$ by $f(t) = e_t$. Each x in the unit ball of $\ell_2[0,1]$ can be written in the form

$$x = \sum_{n=1}^{\infty} \beta_n e_{t_n}$$

where the t_n are distinct numbers in $[0,1]$ and $\sum_{n=1}^{\infty} \beta_n^2 \leq 1$. Accordingly,

$$\langle f,x\rangle = \sum_{n=1}^{\infty} \beta_n \chi_{\{t_n\}} .$$

Hence $x^*f = 0$ except on a countable set for every x^* in $(\ell_2[0,1])^* = \ell_2[0,1]$, and therefore f is scalarly measurable with respect to every Radon probability measure on $[0,1]$. Since $\ell_2[0,1]$ is reflexive, this shows that f is universally Pettis integrable.

Let μ be any Radon probability measure on $[0,1]$, let A be a subset of $[0,1]$ with $\mu(A) > 0$, and let $a < b$. As before, we may assume that $\mu(\{t\}) = 0$ for every t in A . Choose an integer $m > \max\{1/a^2 , 1/b^2\}$ (if either one of a or b is 0, then ignore it in choosing m). With the help of Lemma 9, choose m disjoint measurable subsets $D_1,\ldots,D_m$ of A with $\mu(D_i) > 0$ for each i . Let $x = \sum_{n=1}^{\infty} \beta_n e_{t_n}$ be in the ball of $\ell_2[0,1]$.

Case 1 : $b > 0$. Suppose

$$b < \sup_{t\in D_i} \langle f(t),x\rangle = \sup_{t\in D_i} \sum_{n=1}^{\infty} \beta_n \chi_{\{t_n\}}(t)$$

for each $i=1,\ldots,m$. Then for each i there is some integer n_i for which $\beta_{n_i} > b$. Observe that the integers $n_1,\ldots,n_m$ are distinct since the sets $D_1,\ldots,D_m$ are disjoint. Therefore

$$1 \geq \sum_{n=1}^{\infty} \beta_n^{\,2} \geq \sum_{i=1}^{m} \beta_{n_i}^{\,2} > mb^2 > 1 \ .$$

This contradiction shows that $\sup_{D_i} \langle f,x\rangle \leq b$ for some i .

Case 2 : $b \leq 0$ (hence $a < 0$). Now suppose

$$a > \inf_{t\in D_i} \langle f(t),x\rangle = \inf_{t\in D_i} \sum_{n=1}^{\infty} \beta_n \chi_{\{t_n\}}(t)$$

for each $i=1,\ldots,n$. Then there are distinct integers $n_1,\ldots,n_m$ for which $\beta_{n_i} < a < 0$ for each i . This gives rise to the contradiction

$$1 > \sum_{n=1}^{\infty} \beta_n^{\,2} \geq \sum_{i=1}^{m} \beta_{n_i}^{\,2} > ma^2 > 1$$

thereby showing that $\inf_{D_i} \langle f,x\rangle \geq a$ for some i .

We have thus shown that for each x in the ball of $\ell_2[0,1]$, either $\inf_{D_i} \langle f,x\rangle \geq a$ or $\sup_{D_i} \langle f,x\rangle \leq b$ for one of the sets D_i . Accordingly, the function f has the Bourgain property with respect to μ .

§3 Pettis representable operators and the Bourgain property

Let $S : L_1(\mu) \longrightarrow X^*$ be a bounded linear operator. Recall our separability assumption that guarantees the existence of an increasing sequence (π_n) of finite partitions of Ω that generate the σ-algebra Σ. For each integer n define a function f_n from Ω into X^* by

$$f_n(\cdot) = \sum_{A\in\pi_n} \frac{S(\chi_A)}{\mu(A)} \chi_A(\cdot) .$$

The sequence (f_n,Σ_n) forms a uniformly bounded X^*-valued martingale. Moreover, for each x in the ball of X and for each set A in $\bigcup_{n=1}^{\infty} \Sigma_n$, we have

$$\lim_n \int_A \langle f_n,x\rangle \, d\mu = S(\chi_A)x$$

since the limit is eventually constant. Therefore

$$\lim_n \int_\Omega \langle f_n,x\rangle g \, d\mu = S(g)x$$

for each x in the ball of X and for each g in a dense set, and hence the limit exists for each g in $L_1(\mu)$.

We shall say that the operator S has the Bourgain property if the family $\{\langle f_n,x\rangle : n \in N , ||x|| \leq 1\}$ has the Bourgain property. The main result of this section is the following theorem.

Theorem 12 : An operator $S : L_1(\mu) \longrightarrow X^*$ with the Bourgain property is Pettis representable.

Proof. Because the sequence (f_n) is pointwise uniformly bounded in X^*, we can choose a pointwise weak*-cluster point $f : \Omega \longrightarrow X^*$ of (f_n). Let x be in the unit ball of X. Because $(\langle f_n,x\rangle)$ is a uniformly bounded real-valued martingale, there is a function $h_x : \Omega \longrightarrow R$ such that $\langle f_n,x\rangle$ converges to h_x almost everywhere. However, the function $\langle f,x\rangle$ is a pointwise cluster point of the sequence $(\langle f_n,x\rangle)$ and therefore $\langle f,x\rangle = h_x$ almost everywhere. The Dominated Convergence theorem now ensures that

$$S(g)x = \lim_n \int \langle f_n,x\rangle g \, d\mu = \int \langle f,x\rangle g \, d\mu$$

for all g in $L_1(\mu)$. Hence f is a Gel'fand derivative of S. On the other hand, the function f has the Bourgain property because the family $\{\langle f,x\rangle : ||x|| \leq 1\}$ lies in the pointwise closure of the family $\{\langle f_n,x\rangle : n \in N , ||x|| \leq 1\}$, so that f is Pettis integrable by Theorem 2. This easily implies that f is a Pettis derivative of the operator S and completes the proof.

Theorem 13 : Let $T : X \longrightarrow Y$ be a bounded linear operator for which the set $T(B_X)$ is weakly precompact. If $S : L_1(\mu) \longrightarrow X^*$ is an operator that satisfies $S(\chi_E/\mu(E)) \in T^*(B_{Y^*})$ for every E in Σ of positive measure, then S has the Bourgain property.

<u>Proof</u>. Without loss of generality we may assume that $||T|| \leq 1$. Hence $||S|| \leq 1$. Let (f_n, Σ_n) be the martingale associated with the operator S. We must show that the family $\{<f_n, x> : n \in N, ||x|| \leq 1\}$ has the Bourgain property.

Let $f : \Omega \longrightarrow X^*$ be a pointwise weak*-cluster point of the sequence (f_n) and observe that f takes its values in $T^*(B_{Y^*})$. Suppose that the family $\{<f_n, x> : n \in N, ||x|| \leq 1\}$ fails the Bourgain property. Reread the proof of Theorem 6 to see that there exists a sequence (x_n) in X for which $(<f, x_n>)$ is a copy of the ℓ_1-basis in $L_\infty(\mu)$; that is, there exists $\delta > 0$ such that for all finitely non-zero sequences (a_i) of reals

$$\delta \sum |a_i| \leq ||\sum a_i <f, x_i>||_\infty .$$

But for each ω in Ω, there is some y^* in the ball of Y^* such that $f(\omega) = T^*y^*$, from which it follows that

$$||\sum a_i <f, x_i>||_\infty \leq ||\sum a_i (Tx_i)|| .$$

Therefore (Tx_n) is a copy of the ℓ_1-basis and this contradicts our assumption that $T(B_X)$ is weakly precompact.

<u>Corollary 14</u> : <u>If</u> $S : L_1(\mu) \longrightarrow X^*$ <u>is an operator for which</u> $S^*(B_X)$ <u>is weakly precompact in</u> $L_\infty(\mu)$, <u>then</u> S <u>has the Bourgain property</u>.

Notice that Theorems 12 and 13 combine to give an alternative proof of the implication (a) $\Rightarrow$ (c) in the Factorization Theorem 4.4 .

We do not know if the converse of Theorem 12 is true. An affirmative answer, however, would also give an affirmative answer to the following technical question.

Question : Suppose $S : L_1(\mu) \longrightarrow X^*$ is known to be Pettis representable. Suppose, in addition, that $S(\chi_E/\mu(E))$ belongs to a fixed weak*-compact convex subset K of X^* for each set E of positive measure. Does it follow that S has a Pettis kernel taking its values in K ?

Before leaving this section, we briefly consider the situation for an operator taking values in an arbitrary Banach space X .

Proposition 15 : Let $S : L_1[0,1] \longrightarrow X$ be a bounded linear operator and define the martingale $f_n : [0,1] \longrightarrow X$ by

$$f_n(\cdot) = \sum_{A \in \pi_n} \frac{S(\chi_A)}{\mu(A)} \chi_A(\cdot) .$$

If the family $\{\langle x^*, f_n \rangle : n \in N , ||x^*|| \leq 1\}$ has the Bourgain property, then S is Pettis representable into the second dual X^{**} , and consequently S is a Dunford-Pettis operator into X .

Proof. Let $Q : X \longrightarrow X^{**}$ be the natural embedding of

X into its second dual and set $S_1 = QS : L_1[0,1] \longrightarrow X^{**}$. A moment's reflection shows that the corresponding X^{**}-valued martingale (g_n) for S_1 satisfies $g_n = Qf_n$ for each n . Accordingly, the family $\{<Qf_n,x^*> : n \in N , ||x^*|| \leq 1\} = \{<x^*,f_n> : n \in N , ||x^*|| \leq 1\}$ has the Bourgain property. By Theorem 4, the operator $S_1 = QS$ is Pettis representable, which establishes the first claim. The second claim follows immediately from Observation 4.2 and Stegall's theorem [16, Proposition 3.J] that Pettis representable vector measures on perfect measure spaces have relatively compact ranges.

A partial converse to Proposition 15 fails in the sense that the associated family $\{<x^*,f_n> : n \in N , ||x^*|| \leq 1\}$ for a Dunford-Pettis operator $S : L_1[0,1] \longrightarrow X$ need not have the Bourgain property, as the following example demonstrates.

<u>Example 16</u> : We consider Bourgain's example of a Dunford-Pettis operator into c_o that is not Pettis representable into ℓ_∞ . Accordingly, the associated family of martingales must fail the Bourgain property.

For each integer n , let Π_n denote the finite collection of all possible unions of up to n intervals from the dyadic partition π_n . Let (A_m) be an enumeration of $\bigcup_{n=1}^{\infty} \Pi_n$ and observe that $\lim_m \lambda(A_m) = 0$. Define $S : L_1[0,1] \longrightarrow c_o$ by

$$S\phi = \left(\int_{A_m} \phi \, d\lambda\right)_{m=1}^{\infty}$$

Then S is a Dunford-Pettis operator since the coordinate functions $\chi_{A_m} \in L_\infty[0,1]$ are $L_1[0,1]$-relatively compact (see Theorem 3.10).

Suppose $g : [0,1] \longrightarrow \ell_\infty = c_o^{**}$ is a Gel'fand kernel of S ; that is,

$$S(\phi)x^* = \int <g,x^*>\phi \, d\lambda$$

for each x^* in ℓ_1 and for each ϕ in $L_1[0,1]$. If (e_m) is the usual basis for ℓ_1 , then for each m ,

$$\int_{A_m} \phi \, d\lambda = S(\phi)e_m = \int <g,e_m>\phi \, d\lambda$$

for every ϕ in $L_1[0,1]$. Therefore $\chi_{A_m} = <g,e_m>$ almost everywhere for each integer m .

By the construction of the sets A_m , the sequence (χ_{A_m}) is dense in the space of all $\{0,1\}$-valued functions on $[0,1]$ equipped with the product topology. Let χ_A be a non-measurable pointwise clusterpoint of the sequence (χ_{A_m}) . There exists a subnet (χ_{A_β}) of (χ_{A_m}) such that (χ_{A_β}) converges pointwise to χ_A . Note that $\chi_{A_\beta} = <g,e_\beta>$ almost everywhere for each β . If x^{***} in ℓ_∞^* is a weak*-cluster point of the net (e_β), then we must have $\chi_A = <x^{***},g>$ almost everywhere. Therefore g is not scalarly measurable. Consequently, the operator S can not have a Pettis kernel into ℓ_∞ .

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Vita

Lawrence Hollister Riddle was born March 1, 1954 outside Philadelphia, Pennsylvania. He received a B.S. degree in Mathematics from Carnegie-Mellon University in May, 1976. After spending a year at Cambridge University, England, under the auspices of the Winston Churchill Foundation, he entered the University of Illinois from which he received an M.S. degree in Statistics in May, 1981, and a Ph.D. degree in Mathematics in May. 1982. While at the University of Illinois he received support in the form of teaching assistantships, University of Illinois Graduate Fellowships and a research assistantship from the Graduate Research Board. He is a member of the American Mathematical Society and the Mathematical Association of America.