Approximations in AP Calculus
AP Annual Conference 2006

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Course Description

Derivative at a point
- Tangent line to a curve at a point and local linear approximation
- Approximate rate of change from graphs and tables of values

Applications of derivatives
- Numerical solution of differential equations using Euler's method (BC only)

Numerical approximations to definite integrals Use of Riemann sums (using left, right and midpoint evaluation points) and trapezoidal sums to approximate definite integrals of functions represented algebraically, graphically, and by table of values.

Series of constants (BC only)
- Alternating series with error bound

Taylor series (BC only)
- Lagrange error bound for Taylor polynomials

Tangent Line Approximation (Local Linear Approximation)

Free Response	Multiple Choice
1991 AB3 1995 AB3 1998 AB4 1999 BC6 2002 AB6 (over/under estimate?) 2005 AB6	1969 AB/BC 36 1973 AB 44 1997 AB 14 1998 BC 92

Approximating a Derivative Value

Free Response	Multiple Choice
1998 AB3 (at point in table or from graph) 2001 AB2/BC2 (at point in table) 2003 AB3 (at point not in table) 2005 AB3/BC3 (at point not in table)	

Approximating a Definite Integral

Free Response	Multiple Choice
1994 AB6 (trapezoid from function) 1996 AB3/BC3 (trapezoid from function) 1998 AB3 (midpoint from table) 1999 AB3/BC3 (midpoint from table) 2001 AB2/BC2 (trapezoid from table) 2002(B) AB4/BC4 (trapezoid from graph) 2003 AB3 (left sum from table, unequal widths, over/under estimate?) 2003(B) AB3/BC3 (midpoint from table) 2004(B) AB3/BC3 (midpoint from table) 2005 AB3/BC3 (trapezoid from table, unequal widths) 2006 AB4/BC4 (midpoint from table) 2006(B) AB6 (trapezoid from table, unequal widths)	1973 AB/BC 42 (trapezoid from function) 1988 BC 18 (trapezoid from function) 1993 AB 36 (trapezoid, left from function) 1993 BC 40 (Simpson's rule from function) 1997 AB 89 (trapezoid from table) 1998 AB/BC 9 (estimate from graph) 1998 AB/BC 85 (trapezoid from table, unequal widths) 1998 BC 91 (left from table) 2003 AB/BC 85 (trapezoid from graph, over/under estimate?) 2003 BC 25 (right sum from table, unequal widths)

Error Estimates using Series

Free Response	Multiple Choice
1971 BC4 (alternating series or Lagrange EB) 1976 BC7 (Lagrange EB) 1979 BC4 (alternating series or Lagrange EB) 1982 BC5 (alternating series or Lagrange EB) 1984 BC4 (alternating series) 1990 BC5 (alternating series or Lagrange EB) 1994 BC5 (alternating series) 1999 BC4 (Lagrange EB) 2000 BC3 (alternating series) 2003 BC6 (alternating series) 2004 BC6 (Lagrange EB) 2004(B) BC2 (Lagrange EB) 2006(B) BC6 (alternating series)	

Euler's Method for Differential Equations

Free Response	Multiple Choice
1998 BC4 1999 BC6 (over/under estimate?) 2001 BC5 2002 BC5 2005 BC4 (over/under estimate?) 2006 BC5	2003 BC 5

(Note: 2003(B) means the 2003 Form B exam)

2006 AB–4/BC–4

t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

Rocket A has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \le t \le 80$ seconds, as shown in the table above.

(b) Using correct units, explain the meaning of $\int_{10}^{70} v(t)\, dt$ in terms of the rocket's flight.

Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)\, dt$.

Comments:
Students need to pick out the correct intervals, and the midpoints of those intervals. They should clearly show the setup for their calculation of the midpoint Riemann sum.

A plot of the data suggests that the graph of $v(t)$ is concave down. This is also suggested by the difference quotients between successive data points (since they are decreasing). If that was actually the case, would the midpoint approximation be too small or too large?

2005 AB–3/BC–3

Distance x (cm)	0	1	5	6	8
Temperature $T(x)$ (°C)	100	93	70	62	55

A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature $T(x)$, in degrees Celsius (°C), of the wire x cm from the heated end. The function T is decreasing and twice differentiable.

(a) Estimate $T'(7)$. Show the work that leads to your answer. Indicate units of measure.

(b) Write an integral expression in terms of $T(x)$ for the average temperature of the wire. Estimate the average temperature of the wire using a trapezoidal sum with the four subintervals indicated by the data in the table. Indicate units of measure.

Comments:

It is expected that students should find the "best" estimate for the derivative at $x = 7$. This is a value of x that is not in the table. The best estimate would be obtained by using a symmetric difference quotient using $x = 6$ and $x = 8$. Emphasize the importance of the units.

The subintervals in the table are not of equal length. The trapezoid "rule" cannot be used. Students should either add the areas of the four individual trapezoids, or should average the left and right Riemann sums. It is also important to show the setup for the computations.

2005 BC 4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(c) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(-0.4)$. Show the work that leads to your answer.

(d) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine whether the approximation found in part (c) is less than or greater than $f(-0.4)$. Explain your reasoning.

Comments:

Notice that this Euler's method is going in "backwards" steps, so $\Delta x = -0.2$. Students need experience with doing the computations for both directions.

Over or under approximation is based on the sign of the second derivative over an interval, not just at the starting point. Here $\frac{d^2y}{dx^2} = 2 - 2x + y$. This is positive for $x \le 0$ and $y \ge 0$. Thus the tangent lines will lie underneath the graphs of solution curves in this quadrant.

2004 (Form B) BC2

Let f be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x = 2$ is given by $T(x) = 7 - 9(x-2)^2 - 3(x-2)^3$.

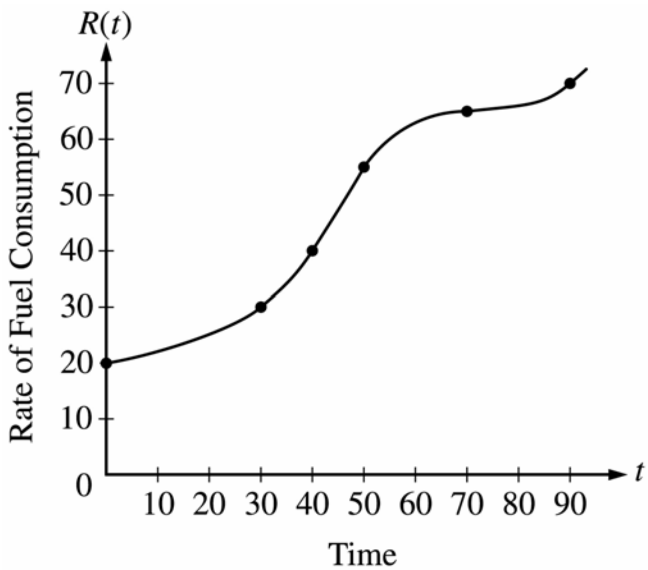
(c) Use $T(x)$ to find an approximation for $f(0)$. Is there enough information given to determine whether f has a critical point at $x = 0$? If not, explain why not. If so, determine whether $f(0)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(d) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 6$ for all x in the closed interval $[0, 2]$. Use the Lagrange error bound on the approximation to $f(0)$ found in part (c) to explain why $f(0)$ is negative.

Comments:

The Lagrange error bound is $\max\limits_{0\le x\le 2}\left|f^{(4)}(x)\right|\dfrac{(2-0)^4}{4!} \le 6\cdot\dfrac{16}{24} = 4$. So $\left|f(0)-T(0)\right| \le 4$.

2003 AB–3



t (minutes)	$R(t)$ (gallons per minute)
0	20
30	30
40	40
50	55
70	65
90	70

The rate of fuel consumption, in gallons per minute, recorded during an airplane flight is given by a twice-differentiable and strictly increasing function R of time t. The graph of R and a table of selected values of $R(t)$, for the time interval $0 \le t \le 90$ minutes, are shown above.

(a) Use data from the table to find an approximation for $R'(45)$. Show the computations that lead to your answer. Indicate units of measure.

(c) Approximate the value of $\int_0^{90} R(t)\,dt$ using a left Riemann sum with the five subintervals indicated by the data in the table. Is this numerical approximation less than the value of $\int_0^{90} R(t)\,dt$? Explain your reasoning.

Comment:
Here is another derivative approximation at a point not in the table. A symmetric difference quotient based on t = 40 and t = 50 gives the best estimate. This is also supported by the graph which shows that the slope of the secant line between 40 and 50 is a good approximation to the tangent line at t = 45.

The subintervals in the table are of unequal lengths so care must be taken when computing the areas of each left rectangle. The over/under approximation for left and right Riemann sums is based on whether the graph is increasing or decreasing.

2002 AB–6

x	–1.5	–1.0	–0.5	0	0.5	1.0	1.5
$f(x)$	–1	–4	–6	–7	–6	–4	–1
$f'(x)$	–7	–5	–3	0	3	5	7

Let f be a function that is differentiable for all real numbers. The table above gives the values of f and its derivative f' for selected points x in the closed interval $-1.5 \le x \le 1.5$. The second derivative of f has the property that $f''(x) > 0$ for $-1.5 \le x \le 1.5$.

(b) Write an equation of the line tangent to the graph of f at the point where $x = 1$. Use this line to approximate the value of $f(1.2)$. Is this approximation greater than or less than the actual value of $f(1.2)$? Give a reason for your answer.

Comments:
The approximation is less than the actual value because of the graph of f is concave up over the <u>entire</u> interval from $x = 1$ to $x = 1.2$. It is not sufficient to check concavity just at $x = 1$. This property is due to the fact that

$$f(b) - \left(f(a) + f'(a)(x-a) \right) = \frac{1}{2} f''(c)(b-a)^2 \text{ for some } c \text{ between } a \text{ and } b$$

or that geometrically, the tangent line lies below the graph when the graph is concave up.

2003 AB 28 (Multiple Choice)

Let g be a twice-differentiable function with $g'(x) > 0$ and $g''(x) > 0$ for all real numbers x, such that $g(4) = 12$ and $g(5) = 18$. Of the following, which is a possible value for $g(6)$?

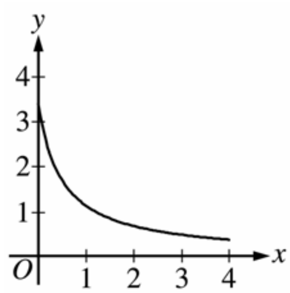
(A) 15 (B) 18 (C) 21 (D) 24 (E) 27

Comments:
This problem can be viewed as an approximation from the Mean Value Theorem. By the MVT, $g(5) - g(4) = g'(c_1) = 6$ for some c_1 between 4 and 5. But using the MVT again, and the fact that $g''(x) > 0$ for all x, we have $g(6) - g(5) = g'(c_2) > g'(c_1) = 6$ for some c_2 between 5 and 6.

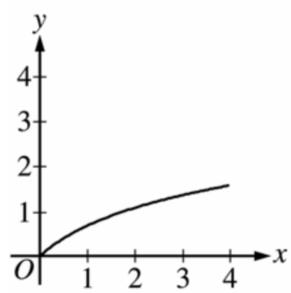
2003 AB/BC 85 (Multiple Choice)

If a trapezoidal sum overapproximates $\int_0^4 f(x)\,dx$, and a right Riemann sum underapproximates $\int_0^4 f(x)\,dx$, which of the following could be the graph of $y = f(x)$?

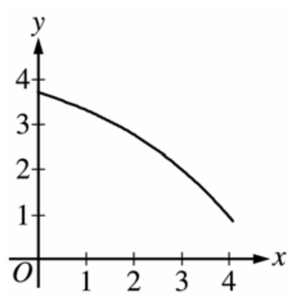
(A)



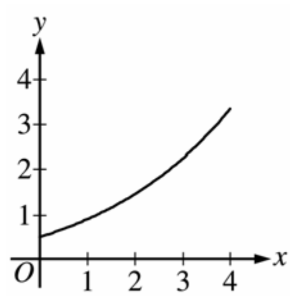
(B)



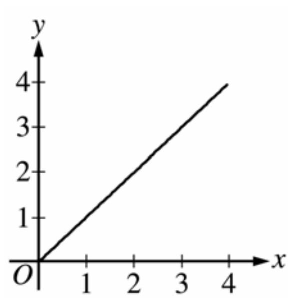
(C)



(D)



(E)



Comments:

If the graph is decreasing, then $\text{Right}(n) < \int_a^b f(x)dx < \text{Left}(n)$.

If the graph is concave down, then $\text{Trap}(n) < \int_a^b f(x)dx < \text{Mid}(n)$

If the graph is increasing or concave up, the respective inequalities are reversed.

Theorem: Let f be a twice differentiable function. Let $T(x) = f(a) + f'(a)(x-a)$ be the tangent line to f at $x = a$. Then

$$f(b) - T(b) = \frac{1}{2} f''(c)(b-a)^2$$

for some value of c between a and b.

Proof:

Define $g(x) = f(x) - T(x) - E(x-a)^2$ where $E = \dfrac{f(b) - T(b)}{(b-a)^2}$. Note that

$$g(a) = f(a) - T(a) = 0$$
$$g(b) = f(b) - T(b) - E(b-a)^2 = 0$$

By the Mean Value Theorem, there is a c_1 between a and b such that $g(c_1) = 0$.

Now $g'(x) = f'(x) - T'(x) - 2E(x-a)$. Note that $g'(a) = f'(a) - T'(a) = 0$. By the Mean Value Theorem applied to $g'(x)$, there is a c_2 between a and c_1 such that $g''(c_2) = 0$. Therefore

$$0 = g''(c_2) = f''(c_2) - T''(c_2) - 2E = f''(c_2) - 2E$$

and so $E = \dfrac{1}{2} f''(c_2)$. Hence $f(b) - T(b) = \dfrac{1}{2} f''(c_2)(b-a)^2$.

Corollary 1: If the graph of f is concave up on the interval (a, b), then the local linear approximation of f at $x = b$ based on the tangent line at $x = a$ is an underestimate.

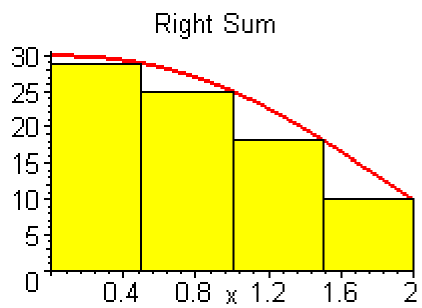
Corollary 2: If $T(x)$ is the tangent line to f at $x = a$, then the error in the local linear approximation of f at $x = b$ using $T(b)$ satisfies

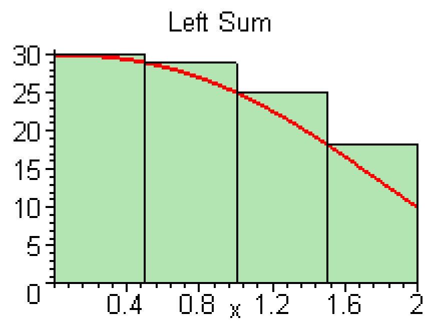
$$|error| = |f(b) - T(b)| \le \max_{a<x<b} |f''(x)| \frac{(b-a)^2}{2}$$

(This is the Lagrange error bound for the Taylor polynomial of f of degree 1 centered at $x = a$.)

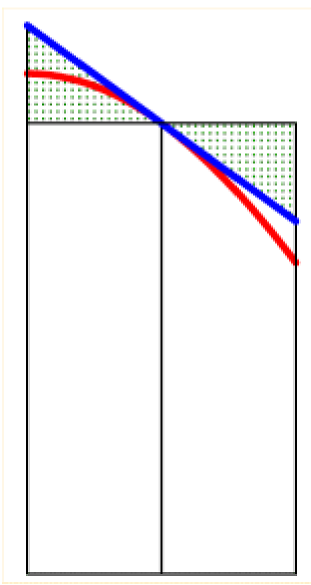
Numerical Approximations of Integrals

If the graph is decreasing, then $\text{Right}(n) < \int_a^b f(x)dx < \text{Left}(n)$.





If the graph is concave down, then $\text{Trap}(n) < \int_a^b f(x)dx < \text{Mid}(n)$



The (blue) line is the tangent line to the graph at the midpoint. This line lies above the graph because the graph is concave down.

The two shaded triangles have equal area.

The area of the midpoint rectangle is therefore equal to the area of the trapezoid with top given by the tangent line.

The area under the graph is therefore less than the area of the midpoint rectangle.

The trapezoid approximation uses the secant line as the top of the trapezoid, and the secant line lies underneath the graph when the graph is concave down.

If the graph is increasing or concave up, the respective inequalities are reversed.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam? (From Student Performance Q&A, AP Central)

2003 AB 3: Students should see more examples of calculating Riemann sums with subintervals of different lengths.

2003 BC 6: Many students said that because the series was alternating, the error for the truncated sum was less than the first omitted term in the series. Teachers need to remind students that the error estimate depends not only on having an alternating series but also on having terms that decrease to 0 in absolute value. Students may lose points on future exams if they do not give or verify all necessary conditions in such situations.

2004 BC 6: Spending a class period comparing the graphs of well-known functions with the graphs of some of their Taylor polynomial approximations would help students see concretely that Lagrange's theorem gives a method to measure the difference between the approximation and the original function. This graphical comparison could serve as the beginning of an analysis and computation of Lagrange error bounds.

2005 AB3/BC3: Help students develop a deeper understanding of estimation methods. Place less emphasis on the trapezoidal rule (the formula) and more emphasis on the trapezoidal method.

2005 BC 4: Teach students how to use concavity to determine the nature of an Euler's method estimate.

2006 AB4/BC4: Have students develop a better understanding of estimation methods for definite integrals. In this problem students had difficulty selecting values for a midpoint Riemann sum.

Calculator Commands for Riemann Sums

		TI-83/84	**TI-85/86**	**TI-89**
Definite Integrals	`fnInt( f(x), x, lower limit, upper limit)`			
	fnInt	MATH 9: fnInt	CALC fnInt	F3 B: nInt
Use function from Y=	To use a function entered as Y_1 in some other command (like fnInt or seq)			
	Y_1	VARS Y-VARS 1: Function 1:Y_1	Type y1 using 2nd-alpha for lowercase y	Type y1(x) from keyboard
Riemann Sums	`sum( seq( f(x), x, first, last, Δx))* Δx`			
	sum	LIST MATH 5: sum	LIST OPS MORE sum	MATH 3: List 6: sum
	seq	LIST OPS 5: seq	LIST OPS MORE seq	MATH 3: List 1: seq

Symbolic Riemann Sums
 	
$\int_a^b f(x)\,dx$
 	
***f*(x) a polynomial**
 	
(TI-89 only)
	Left:
	`Σ( f(a+(k-1)*dx), k, 1, n)*dx | n = (b-a)/dx`
	Right:
	`Σ( f(a+k*dx), k, 1, n)*dx | n = (b-a)/dx`
	Midpoint:
	`Σ( f(a+(k-1/2)*dx), k, 1, n)*dx | n = (b-a)/dx`
	
	`expand(ans(1))`			

	Σ			F3 calc 4: Σ(sum
	expand			F2 algebra 3: expand(

Numerical Integration Worksheet

$$\int_0^2 x^4 - 3x^3 - 3x^2 + 30\,dx$$

n	Left sum	error	ratio	n	Right sum	error	ratio
2			XXXXXX	2			XXXXXX
4				4			
8				8			
16				16			
32				32			

n	Midpoint	error	ratio	n	Trapezoid	error	ratio
2			XXXXXX	2			XXXXXX
4				4			
8				8			
16				16			
32				32			

error = approximation – integral

"ratio" means take the error for n = 2 and divide by the error for n = 4. Put this in the box for n = 4. Then take the error for n = 4 and divide by the error for n = 8. Record this in the box for n = 8. Divide the error for n = 8 by the error for n = 16. Finally, take the error for n = 16 and divide by the error for n = 32.

Numerical Integration Worksheet

$\int_0^2 x^4 - 3x^3 - 3x^2 + 30\,dx$ **= 46.4**

n	Left sum	error	ratio	n	Right sum	error	ratio
2	**55**	**8.6**	XXXXXX	2	**35**	**–11.4**	XXXXXX
4	**51.0625**	**4.6625**	**1.84**	4	**41.0625**	**–5.3375**	**2.14**
8	**48.81641**	**2.41641**	**1.93**	8	**43.81641**	**–2.58359**	**2.07**
16	**47.62915**	**1.22915**	**1.97**	16	**45.12915**	**–1.27085**	**2.03**
32	**47.01979**	**0.61979**	**1.98**	32	**45.76979**	**–0.63021**	**2.02**

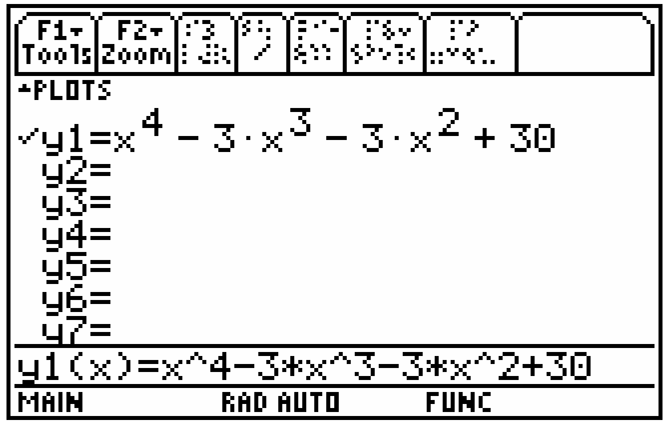
n	Midpoint	error	ratio	n	Trapezoid	error	ratio
2	**47.125**	**0.725**	XXXXXX	2	**45**	**–1.4**	XXXXXX
4	**46.57031**	**0.17031**	**4.26**	4	**46.0625**	**–0.33755**	**4.15**
8	**46.44189**	**0.04189**	**4.07**	8	**46.31641**	**–0.08359**	**4.04**
16	**46.41043**	**0.01043**	**4.02**	16	**46.37915**	**–0.02085**	**4.02**
32	**46.40261**	**0.00261**	**4.00**	32	**46.39479**	**–0.00521**	**4.00**

error = approximation – integral

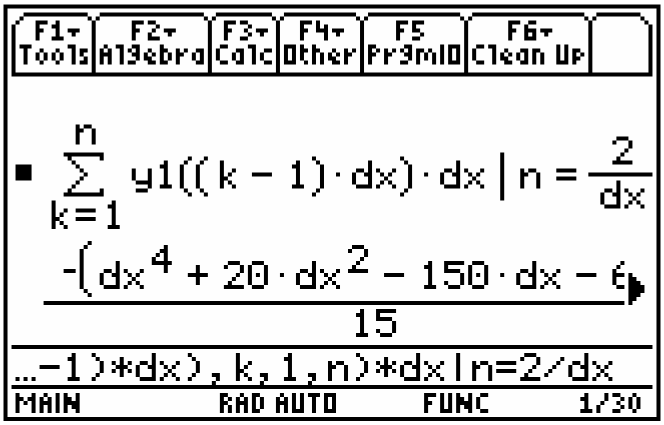
"ratio" means take the error for n = 2 and divide by the error for n = 4. Put this in the box for n = 4. Then take the error for n = 4 and divide by the error for n = 8. Record this in the box for n = 8. Divide the error for n = 8 by the error for n = 16. Finally, take the error for n = 16 and divide by the error for n = 32.

Symbolic Riemann Sums on the TI-89

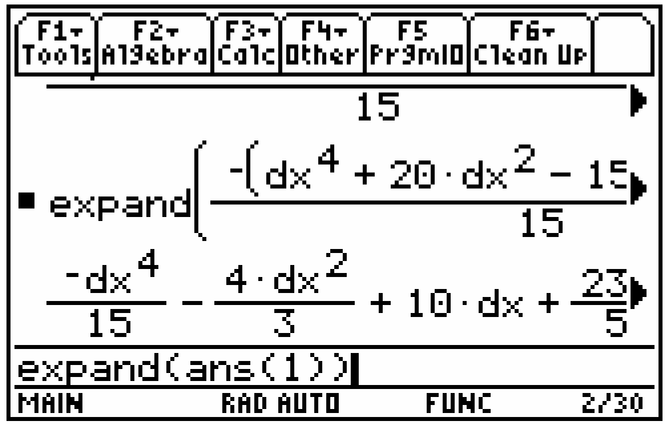
$$\int_0^2 x^4 - 3x^3 - 3x^2 + 30\,dx$$



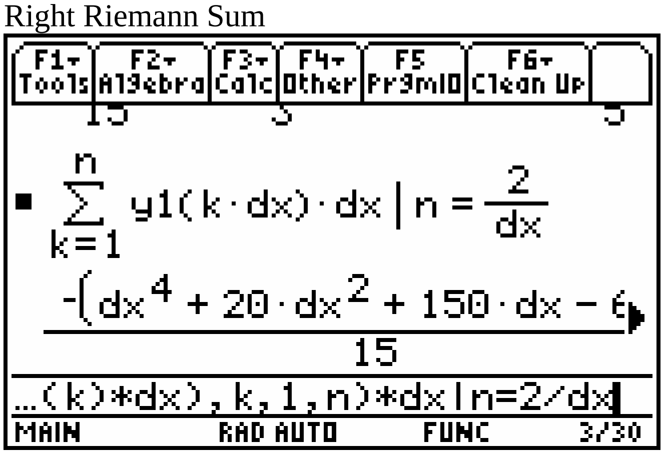
Left Riemann Sum



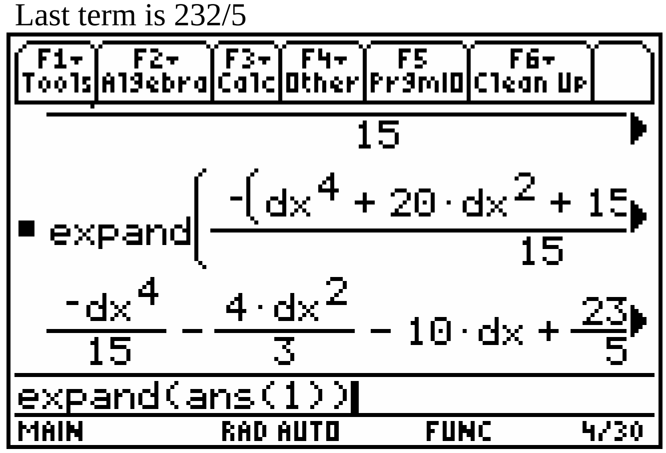
Last term is 232/5



Right Riemann Sum



Last term is 232/5



Midpoint Riemann Sum